ILL-POSEDNESS AND STABILITY ESTIMATE FOR THE HEAT EQUATION BACKWARD IN TIME WITH DIRICHLET AND INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT

In this paper, we first prove that the heat equation backward in time with Dirichlet and integral boundary conditions is an ill-posed problem. Then, we establish a stability estimate of Hölder type for this ill-posed problem.

Keywords: Heat equation backward; ill-posed problem; stability estimate.

1. Introduction

Let l and T be positive real numbers. Consider the problem of determining $u(x,t)$ satisfying

$$\begin{cases} u_l - u_{xx} = 0, 0 < x < l, 0 < t < T \\ u(0,t) = 0,\ \int_0^l u(x,t)dx = 0,\ \ 0 \le t \le T \end{cases} \tag{1}$$

with measurement data at $t = T$:

$$u(x,T) = \varphi(x),\ x \in [0,l] \tag{2}$$

where φ is a given function.

Problem (1)-(2) is ill-posed in the sense of Hadamard (see Theorem 2.1). Therefore, stability estimates and regularization methods are desired.

In recent years, there have been many research papers on parabolic equations with integration conditions ([1], [2], [3], [4], [5], [6], [7]). In paper [1], we have established a stability estimate result of the Hölder type for heat equation backward in time with Neumann and integral boundary conditions. However, the technique in paper [1] cannot be used to establish stability estimate results for heat equation backward in time with Dirichlet and integral boundary conditions because the two boundary conditions, Dirichlet and Neumann, are different. Furthermore, to our knowledge, there is so far not any result on stability estimates of heat equation backward in time with Dirichlet and integral bound-ary conditions (1)-(2). This motivates us to establish a stability estimate result for problem (1)-(2).

The purpose of this paper is to prove that problem (1)-(2) is an ill-posed problem and propose a stability estimate result of the Hölder type for this problem. We present these results in Section 2.

2 Main results

For simplicity of notation, in this section we denote $\|\cdot\|_{L^2(0,l)}$ by $\|\cdot\|$.

Theorem 2.1. (ill-posedness) *Problem* (1)-(2) *is an ill-posed problem.*

Proof. Set

$$\begin{aligned}
&u^n(x,t) = \frac{1}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \sin \frac{2\pi n x}{l},\ 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T,\\
&\varphi^n(x) = \frac{1}{n} \sin \frac{2\pi n x}{l},\ 0 \leqslant x \leqslant l,\\
&u(x,t) = 0,\ 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T,\\
&\varphi(x) = 0,\ 0 \leqslant x \leqslant l.
\end{aligned}$$

It follows that

$$\begin{aligned}
&u^n(0,t) = 0,\ 0 \leqslant t \leqslant T,\\
&u^n_t(x,t) = -\left(\frac{2\pi n}{l}\right)^2 \frac{1}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \sin \frac{2\pi n x}{l},\ 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T,\\
&u^n_{xx}(x,t) = \frac{2\pi n}{l} \frac{1}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \cos \frac{2\pi n x}{l},\ 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T,\\
&u^n_{xx}(x,t) = -\left(\frac{2\pi n}{l}\right)^2 \frac{1}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \sin \frac{2\pi n x}{l},\ 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T,
\end{aligned}$$

$$\begin{aligned}
\int_0^l u^n(x,t)dx &= \int_0^l \frac{1}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \sin \frac{2\pi n x}{l} dx\\
&= \frac{1}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \int_0^l \sin \frac{2\pi n x}{l} dx\\
&= -\frac{1}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \frac{l}{2\pi n} \cos \frac{2\pi n x}{l}\Big|_0^l\\
&= -\frac{l}{n} e^{\left(\frac{2\pi n}{l}\right)^2 (T-t)} \frac{\cos 2\pi n - \cos 0}{2\pi n} = 0,\ 0 \leqslant t \leqslant T,\\
u^n(x,T) &= \frac{1}{n} \sin \frac{2\pi n x}{l} = \varphi^n(x),\ 0 \leqslant x \leqslant l,
\end{aligned}$$

$$\begin{aligned}
&u_t(x,t) = 0 = u_{xx}(x,t),\ 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T,\\
&u(0,t) = 0,\ \int_0^l u(x,t)dx = 0,\ 0 \leqslant t \leqslant T,\\
&u(x,T) = 0 = \varphi(x),\ 0 \leqslant x \leqslant l.
\end{aligned}$$

Therefore $u^n(x,t)$ satisfies the following conditions

$$\begin{cases}
u_t^n - u_{xx}^n = 0,\ 0 < x < l, 0 < t < T\\
u^n(0,t) = 0,\ \int_0^l u^n(x,t)dx = 0, 0 \le t \le T\\
u^n(x,T) = \varphi^n(x),\ 0 \leqslant x \leqslant l
\end{cases}$$

and $u(x,t)$ satisfies the following conditions

$$\begin{cases}
u_t - u_{xx} = 0,\ 0 < x < l, 0 < t < T\\
u(0,t) = 0,\ \int_0^l u(x,t)dx = 0, 0 \le t \le T\\
u(x,T) = \varphi(x),\ 0 \leqslant x \leqslant l.
\end{cases}$$

On the other hand, we have

$$\begin{aligned}
\|\varphi^n - \varphi\| &= \left(\int_0^l \left(\varphi^n(x)\right)^2 dx\right)^{\frac{1}{2}} = \left(\int_0^l \left(\frac{1}{n}\sin\frac{2\pi n x}{l}\right)^2 dx\right)^{\frac{1}{2}}\\
&= \frac{1}{n}\left(\int_0^l \sin^2\frac{2\pi n x}{l})dx\right)^{\frac{1}{2}} = \frac{1}{n}\left(\int_0^l \frac{1-\cos\dfrac{4\pi n x}{l}}{2}dx\right)^{\frac{1}{2}}\\
&= \frac{1}{n}\left(\frac{1}{2}x\Big|_0^l - \frac{l\sin\dfrac{4\pi n x}{l}}{8\pi n}\Big|_0^l\right)^{\frac{1}{2}}\\
&= \frac{\sqrt{l}}{\sqrt{2}n} \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

However, for $0 \leqslant t < T$ we obtain

$$\begin{aligned}
\|u^n(\cdot,t) - u(\cdot,t)\| &= \left(\int_0^l u^n(x,t)^2(x)dx\right)^{\frac{1}{2}} = \left(\int_0^l \left(\frac{1}{n}e^{\left(\frac{2\pi n}{l}\right)^2(T-t)}\sin\frac{2\pi n x}{l}\right)^2 (x)dx\right)^{\frac{1}{2}}\\
&= \frac{1}{n}e^{\left(\frac{2\pi n}{l}\right)^2(T-t)}\left(\int_0^l \left(\sin\frac{2\pi n x}{l}\right)^2 (x)dx\right)^{\frac{1}{2}}\\
&= \frac{\sqrt{l}}{\sqrt{2}n}e^{\left(\frac{2\pi n}{l}\right)^2(T-t)} \to +\infty \text{ as } n \to +\infty.
\end{aligned}$$

This proves that problem (1)-(2) is an ill-posed problem.
Hence, the theorem is proved. □

Theorem 2.2. (Stability estimate) *Let $u^1(x,t)$ and $u^2(x,t)$ be solutions of problem* (1)*. Assume that there exists a constant $C \leq 0$ such that*

$$u^i(l,t) = Cu^i_x(0,t), i = 1,2. \tag{3}$$

If $u^1(x,t)$ and $u^2(x,t)$ satisfy

$$\|u^i(\cdot,0)\| \leqslant E,\ i = 1,2. \tag{4}$$

and $\|u^1(\cdot,T) - u^2(\cdot,T)\| \leqslant \delta$ where δ and E are some positive constants then

$$\|u^1(\cdot,t) - u^2(\cdot,t)\| \leqslant 2^{\left(1-\frac{t}{T}\right)}\delta^{\frac{t}{T}}E^{1-\frac{t}{T}},\ \forall t \in [0,T].$$

Proof. Set

$$u(x,t) = u^1(x,t) - u^2(x,t),\ (x,t) \in [0,l] \times [0,T].$$

We have

$$\begin{aligned} u_t(x,t) &= u^1_t(x,t) - u^2_t(x,t) \\ &= u^1_{xx}(x,t) - u^2_{xx}(x,t) \\ &= u_{xx}(x,t), 0 < x < l, 0 < t < T \\ u(0,t) &= u^1(0,t) - u^2(0,t) = 0 \\ \int_0^l u(x,t)dx &= \int_0^l \left(u^1(x,t) - u^2(x,t)\right) dx = 0, 0 \leq t \leq T. \end{aligned}$$

Thus, the function $u(x,t)$ satisfies the following conditions

$$\begin{cases} u_t - u_{xx} = 0,\ 0 < x < l, 0 < t < T \\ u(0,t) = 0, \int_0^l u(x,t)dx = 0, 0 \leq t \leq T. \end{cases} \tag{5}$$

Set $h(t) = \int_0^l (u(x,t))^2\, dx = \|u(\cdot,t)\|^2,\ t \in [0,T]$. By the integration by part, we have

$$\begin{aligned} h'(t) &= 2\int_0^l u(x,t)u_t(x,t)dx \\ &= 2\int_0^l u(x,t)u_{xx}(x,t)dx \\ &= 2\int_0^l u(x,t)d(u_x(x,t)) = 2u(x,t)u_x(x,t)\Big|_0^l - 2\int_0^l (u_x(x,t))^2\, dx \\ &= 2u(l,t)u_x(l,t) - 2u(0,t)u_x(0,t) - 2\int_0^l (u_x(x,t))^2\, dx \\ &= 2u(l,t)u_x(l,t) - 2\int_0^l (u_x(x,t))^2\, dx \text{ (since } u(0,t) = 0) \end{aligned}$$

$$\begin{aligned}
h''(t) &= 2u_t(l,t)u_x(l,t) + 2u(l,t)u_{xt}(l,t) - 4\int_0^l u_x(x,t)u_{xt}(x,t)dx \\
&= 2u_t(l,t)u_x(l,t) + 2u(l,t)u_{xt}(l,t) - 4u_x(x,t)u_t(x,t)\Big|_0^l + 4\int_0^l (u_t(x,t))^2\,dx \\
&= 2u_t(l,t)u_x(l,t) + 2u(l,t)u_{xt}(l,t) - 4u_x(l,t)u_t(l,t) \\
&\quad + 4\int_0^l (u_t(x,t))^2\,dx, \text{(since } u(0,t)=0\ ,\ u_t(0,t)=0) \\
&= 2u(l,t)u_{xt}(l,t) - 2u_x(l,t)u_t(l,t) + 4\int_0^l (u_t(x,t))^2\,dx. \qquad (6)
\end{aligned}$$

From the assumption

$$u^i(l,t) = Cu_x^i(0,t), i = 1,2 \qquad (7)$$

we have

$$\begin{cases} u^1(l,t) = Cu_x^1(0,t) \\ u^2(l,t) = Cu_x^2(0,t). \end{cases} \qquad (8)$$

This implies that

$$Cu_x(0,t) = Cu_x^1(0,t) - Cu_x^2(0,t) = u^1(l,t) - u^2(l,t) = u(l,t). \qquad (9)$$

It follows from

$$\int_0^l u(x,t)dx = 0$$

that

$$\int_0^l u_t(x,t)dx = 0.$$

This implies that

$$\int_0^l u_{xx}(x,t)dx = 0 \text{ (since } u_t(x,t) = u_{xx}(x,t)).$$

Therefore, we obtain

$$u_x(l,t) = u_x(0,t). \qquad (10)$$

From (9) and (10), we have

$$u(l,t) = Cu_x(l,t).$$

This implies that

$$\begin{aligned} h'(t) &= 2u(l,t)u_x(l,t) - 2\int_0^l (u_x(x,t))^2\,dx \\ &= 2Cu_x^2(l,t) - 2\int_0^l (u_x(x,t))^2\,dx \le 0 \text{ (since } C \le 0), \forall t \in (0,T). \end{aligned}$$

If $C < 0$, then from the above equality, we have

$$\begin{aligned} &2u(l,t)u_{xt}(l,t) - 2u_x(l,t)u_t(l,t) \\ &\qquad = \frac{2}{C}u(l,t)u_t(l,t) - \frac{2}{C}u(l,t)u_t(l,t) \\ &\qquad = 0. \end{aligned} \tag{11}$$

If $C = 0$ then $u(l,t) = 0$. This implies that $u_t(l,t) = 0$. Therefore, we get

$$2u(l,t)u_{xt}(l,t) - 2u_x(l,t)u_t(l,t) = 0. \tag{12}$$

From (6), (11) and (12), we have

$$h''(t) = 4\int_0^l (u_t(x,t))^2\,dx. \tag{13}$$

Due to the Cauchy-Schwarz inequality, we have

$$\left(\int_0^l u(x,t)u_t(x,t)dx\right)^2 \leqslant \int_0^l (u(x,t))^2\,dx \int_0^l (u_t(x,t))^2\,dx.$$

This implies that

$$\begin{aligned} &h(t)h''(t) - (h'(t))^2 \\ &\qquad = 4\left(\int_0^l (u(x,t))^2\,dx \int_0^l (u_t(x,t))^2\,dx - \left(\int_0^l u(x,t)u_t(x,t)dx\right)^2\right) \\ &\qquad \geqslant 0,\ \forall t \in [0,T]. \end{aligned}$$

This proves that f is a logarithmically convex function. Therefore, we have

$$h(t) \leqslant h(T)^{\frac{t}{T}} h(0)^{1-\frac{t}{T}},\ \forall t \in [0,T]$$

or

$$\|u(\cdot,t)\| \leqslant \|u(\cdot,T)\|^{\frac{t}{T}} \|u(\cdot,0)\|^{1-\frac{t}{T}},\ \forall t \in [0,T]. \tag{14}$$

On the other hand, we have

$$\|u(\cdot,T)\| = \|u^1(\cdot,T) - u^2(\cdot,T)\| \leqslant \delta \tag{15}$$

and

$$\|u(\cdot,0)\| = \|u^1(\cdot,0) - u^2(\cdot,0)\| \leqslant \|u^1(\cdot,0)\| + \|u^2(\cdot,0)\| \leqslant E + E = 2E. \tag{16}$$

From (14), (15) and (16), we obtain

$$\|u^1(\cdot,t) - u^2(\cdot,t)\| \leqslant 2^{\left(1-\frac{t}{T}\right)}\delta^{\frac{t}{T}}E^{1-\frac{t}{T}},\ \forall t \in [0,T].$$

Hence, the theorem is proved. □

Next, we will take an example of specific functions $u^1(x,t)$, $u^2(x,t)$ satisfying the assumptions of Theorem 2.2.

Example 2.3. Let l, T, δ and E be the positive constants mentioned in (1) and in Theorem 2.2. Choose n_0 to be a positive integer satisfying

$$n_0 \geq \max\left\{\frac{\sqrt{l}}{E}, \frac{l}{2\pi}\sqrt{\frac{1}{T}\ln\frac{l}{\delta^2}}\right\}. \tag{17}$$

Consider the functions

$$u^1(x,t) = \frac{1}{n_0}e^{-\left(\frac{2\pi n_0}{l}\right)^2 t}\sin\frac{2\pi n_0 x}{l}, 0 \leq x \leq l, 0 \leq t \leq T \tag{18}$$

$$u^2(x,t) = 0,\ 0 \leq x \leq l, 0 \leq t \leq T. \tag{19}$$

It is easy to check that for $i = 1, 2$

$$\begin{cases} u^i_t - u^i_{xx} = 0,\ 0 < x < l, 0 < t < T \\ u^i(0,t) = 0,\ \int_0^l u^i(x,t)dx = 0, 0 \leq t \leq T. \end{cases} \tag{20}$$

Furthermore

$$u^1_x(x,t) = \frac{2\pi}{l}e^{-\left(\frac{2\pi n_0}{l}\right)^2 t}\cos\frac{2\pi n_0 x}{l}$$
$$u^2_x(x,t) = 0.$$

This implies that

$$u^1_x(0,t) = \frac{2\pi}{l}e^{-\left(\frac{2\pi n_0}{l}\right)^2 t}$$
$$u^2_x(0,t) = 0.$$

It follows from $u^1(l,t) = u^2(l,t) = 0$ that

$$u^i(l,t) = Cu^i_x(0,t), i = 1, 2 \text{ with } C = 0.$$

Now we evaluate $\|u^i(\cdot,0)\|, i=1,2$. It is clear that $\|u^2(\cdot,0)\|=0<E$. We have

$$\begin{aligned}
\|u^1(\cdot,0)\|^2 &= \int_0^l \left(u^1(x,0)\right)^2 dx\\
&= \int_0^l \left(\frac{1}{n_0}\right)^2 \sin^2\frac{2\pi n_0 x}{l}dx\\
&= \left(\frac{1}{n_0}\right)^2 \int_0^l \sin^2\frac{2\pi n_0 x}{l}dx\\
&\leqslant \left(\frac{1}{n_0}\right)^2 \int_0^l dx = \frac{l}{n_0^2}\\
&\leqslant E^2 \text{ (since } n_0 \geqslant \frac{\sqrt{l}}{E}).
\end{aligned}$$

Therefore, we obtain $\|u^1(\cdot,0)\| \leqslant E$. Next we evaluate $\|u^1(\cdot,T)-u^2(\cdot,T)\|$. We have

$$\begin{aligned}
\|u^1(\cdot,T)-u^2(\cdot,T)\|^2 &= \int_0^l \left(u^1(x,T)-u^2(x,T)\right)^2 dx\\
&= \int_0^l \frac{1}{n_0^2}e^{-\left(\frac{2\pi n_0}{l}\right)^2 T}\sin^2\frac{2\pi n_0 x}{l}dx\\
&\leq \frac{1}{n_0^2}e^{-\left(\frac{2\pi n_0}{l}\right)^2 T}\int_0^l \sin^2\frac{2\pi n_0 x}{l}dx\\
&\leq \frac{1}{n_0^2}e^{-\left(\frac{2\pi n_0}{l}\right)^2 T}\int_0^l dx\\
&= \frac{1}{n_0^2}e^{-\left(\frac{2\pi n_0}{l}\right)^2 T} l\\
&\leq e^{-\left(\frac{2\pi n_0}{l}\right)^2 T} l\\
&\leq \delta^2 \text{ (since } n_0 \geq \frac{l}{2\pi}\sqrt{\frac{1}{T}\ln\frac{l}{\delta^2}}).
\end{aligned}$$

This implies that $\|u^1(\cdot,T)-u^2(\cdot,T)\| \leqslant \delta$.
Thus, the functions $u^1(x,t)$ and $u^2(x,t)$ satisfy the assumptions of Theorem 2.2. Therefore, we have

$$\|u^1(\cdot,t)-u^2(\cdot,t)\| \leqslant 2^{\left(1-\frac{t}{T}\right)}\delta^{\frac{t}{T}}E^{1-\frac{t}{T}},\ \forall t\in[0,T].$$

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TÓM TẮT

TÍNH ĐẶT KHÔNG CHỈNH VÀ ĐÁNH GIÁ ỔN ĐỊNH CHO PHƯƠNG TRÌNH TRUYỀN NHIỆT NGƯỢC THỜI GIAN VỚI CÁC ĐIỀU KIỆN BIÊN DIRICHLET VÀ TÍCH PHÂN

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Ngày nhận bài 05/8/2024, ngày nhận đăng 17/10/2024

Trong bài báo này, đầu tiên chúng tôi chứng minh phương trình truyền nhiệt ngược thời gian với các điều kiện biên Dirichlet và tích phân là một bài toán đặt không chỉnh. Sau đó, chúng tôi thành lập kết quả đánh giá ổn định kiểu Hölder cho bài toán này.

Từ khóa: Phương trình truyệt nhiệt ngược; bài toán đặt không chỉnh; đánh giá ổn định.