STABILITY AND STABILIZATION USING DISCRETE-TIME FEEDBACK CONTROL FOR HYBRID STOCHASTIC DELAY SYSTEMS WITH GENERAL DELAY*

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Abstract. This paper develops stability and stabilization for hybrid stochastic differential delay equations (SDDEs) with general time-variable delays using the Halanay-type inequalities. First, a right-continuous version of the Halanay inequality is proved using methods different from the usual approach of proof by contradiction method. Because the Halanay inequality does not require much on the delay, boundedness and stability in mean square of hybrid SDDEs with general time delays are obtained. For stability not focusing on the equilibrium points, asymptotic stability in distribution is an appropriate criterion and has been studied extensively. For this type of stability, it is crucial to use time-homogeneous Markov processes. In this work, the problem is examined by treating delays that behave periodically. The proof is conducted using probabilistic argument, segment processes, and weaker conditions than that of the moment conditions. For a given hybrid SDDE being unstable in distribution, a feedback control based on discrete-time observations is constructed to stabilize the underlying system. For the controlled systems (hybrid SDDEs with non-constant delays), time-homogeneous Markov processes are also identified. The use of Halanay inequality enables us to obtain the upper bound of observation duration using linear equations rather than the cumbersome exponential equations. Finally, two examples are given to demonstrate the effectiveness of our theory. It is shown that a better bound of observation duration can be achieved compared with the existing results.

Key words. Hybrid stochastic differential delay equation, time-variable delay, Halanay inequality, stability in distribution, discrete-time feedback control.

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1. Introduction.

Stochastic differential delay equations with Markovian switching (the so-called hybrid SDDEs) have been used as a useful modeling tool for numerous practical systems. The hybrid systems are highlighted by the coexistence of continuous dynamics and discrete events, which are used to model environmental noises, abrupt changes in structures or parameters, and the influence of past history. Stability and stabilization have received heightened attention; see [11, 15, 16, 18, 23, 27, 34, 36] and references therein. In the traditional Lyapunov analysis, to tackle the delay effect, the delay function $\delta(t) : [0, \infty) \to [0, h]$ is required to be differentiable with its derivative being less than 1 (e.g., [16, 24, 31]), but this leaves many real-world delays excluded such as the piecewise constant delay [8] and saw-tooth delay [28]. To remove this restriction, effort has been devoted to a relatively weaker condition based on Lebesgue measure theory; see [4, 15],

$$\limsup_{\Delta \to 0^+} \left(\sup_{s \geq -h} \frac{\text{Leb}(\{t \geq 0 : t - \delta(t) \in [s, s + \Delta)\})}{\Delta} \right) < \infty. \tag{1.1}$$

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Although this has alleviated some restrictions, one may still wonder what if our delay functions violate condition (1.1) such as the commonly seen sampling-holding delay $\delta(t) = t - [t/h]h$ (e.g., [14]). Is it possible to study such delays and is there any appropriate tool available?

The answer is positive. Some powerful tools have been developed such as the Razumikhin method (RM) [26] and the Halanay inequality (HI) [9]. [Recall that HI is a comparison theorem for differential delay equations, which has been used to analyze stability of delay equations. For a nonnegative real-valued function $v(\cdot)$, HI states that if

$$\dot{v}(t) \le -\lambda_2 v(t) + \lambda_3 \sup_{-h\le\theta\le 0} v(t+\theta) \quad \text{with} \quad \lambda_2 > \lambda_3 \ge 0, \tag{1.2}$$

then $v(t) \le k\exp(-\eta(t-t_0))$ for $t \ge t_0$, where $k > 0$ and $\eta > 0$.] The basic idea of these two methods are similar: if at some time t, state $v(t)$ reaches the supremum of the whole past segment, then the derivative $\dot{v}(t)$ becomes negative whence the state decreases; otherwise, current state could not exceed the history and behave the tendency of decay. Because it only operates in a pointwise setting, it does not require much on the information of delay functions. There are effective approaches for dealing with asymptotic behaviors of delay systems (e.g., see [10, 23, 33, 35] for RM, and [1, 3, 13, 25, 32, 40] for HI). RM indeed works well, but compared with HI, it is somewhat inconvenient to verify the conditions, and to provide qualitative properties rather than quantitative estimates for system states. Therefore, in this paper, we will use HI to study the stability of hybrid SDDEs with more general delays.

In addition, not only are we concerned with asymptotic stability of equilibrium state (ASE), but also we deal with asymptotic stability in distribution (ASD). Compared with the former, ASD is in fact more appropriate for many real stochastic systems since the equilibrium states do not always exist (e.g., the mean reverting processes [20]). In many practical systems, we expect that the solutions would converge in the sense of in probability distribution rather than to a steady state, especially for population systems under random environment, where coexistence of species is more desired than extinction [12]. Though some results have been achieved in the past two decades (see [2, 5, 30, 38, 39]), ASD has not been extensively studied compared to ASE due to much more complex mathematics involved. Moreover, for hybrid SDDEs, the existing papers (e.g., [5, 39]) mainly considered the delay function to be a constant, not time-varying, or under condition (1.1), or more general condition. To the best of our knowledge, there has not been much work on using HI to study ASD of hybrid SDDEs with time-variable delays.

Furthermore, it is highly desirable to use feedback control to stabilize systems. Traditionally, control is designed based on continuous-time state observations. Nevertheless, it has been recognized that it is more reasonable and less costly in practice if observations are made at discrete epochs $0, \tau, 2\tau, \dots$ (see [21]). In the past few year, the literature on stabilization by discrete-time feedback control has been growing rapidly (e.g., [6, 22, 35, 37]). However, these papers have paid much attention to the stabilization of equilibrium, and up to now, only [17, 19] considered the discrete-state-feedback stabilization in the sense of ASD. But the original unstable systems investigated were all delay-free.

In this paper, we use HI to study the stability and stabilization of hybrid SDDEs with general time-variable delays. We first treat boundedness and ASE. Then if equilibria are not the primary concerns or information on equilibrium points is not available, we concentrate on the weaker property ASD. To stabilize unstable hybrid

SDDEs in the sense of ASD, the design of feedback control based on discrete-time observations is proposed. Compared with the existing results, the main contributions and highlights of this paper are given as follows.

(1) In contrast to the usual continuity condition, the HI is developed under right continuity. In [1, 25], HI (1.2) was generalized from constants λ_2, λ_3 to functions $\lambda_2(t), \lambda_3(t)$, and further developed to $\lambda_1 - \lambda_2(t)v(t) + \lambda_3(t)\sup_{-h\le\theta\le 0} v(t+\theta)$ in [3, 13, 32], so that the boundedness of the systems can be derived. Note that it is often required the right-hand side of (1.2) being continuous (including $v(t), \lambda_2(t), \lambda_3(t)$). However, for our hybrid SDDEs, $v(t)$ will always be used in the form of $E|x(t)|^p$. Although the solution $x(t)$ is continuous, its pth-moment may not be; see [35]. In this case, right-continuity is more appropriate than straight continuity, especially for systems with Markovian switching. It should be emphasized that our work is not a straightforward generalization of that under the continuity assumption. Note also that the contradiction method used in the proof (e.g., [3, 25]) is not valid when $v(t)$ only has right-continuity. As an alternative, we present novel idea to prove the boundedness and stability of right-continuous version of HI.

(2) For ASD, delay functions of hybrid SDDEs are non-constant. In the study of ASD, time homogeneity is of critical importance. For delay equations, the solution process $x(t)$ does not possess Markov property, so we need to focus on the segment process $x_t = \{x(t+\theta) : -h \le \theta \le 0\}$. Unfortunately, x_t is usually not time-homogeneous since $\delta(t)$ is a function of time, and shifting the time axis would give us a different hybrid SDDE. As a result, there is virtually no results for ASD of non-constant time delays. Recently, [2, 19] explained how to find time-homogeneous Markov segment processes for periodic non-delay systems. Motivated by their work, we consider our time-variable delays to be periodic and present proper time-homogeneous Markov processes. The difference is however, we will prove ASD under the probability conditions of segment processes rather than the stronger moment conditions (see Definition 4.3 below).

(3) Our discrete-feedback controlled system is a hybrid SDDE with two periodic delays. How to select τ and form segment processes wisely is crucial for the underlying Markov processes to be time-homogeneous. Moreover, in the proof of ASD for controlled systems, [17, 19] both adopted the exponential type of integral inequalities, and hence the bound of τ was given by solving exponential equations. In contrast, we will apply HI to show ASD, where the delay is dealt with under the worst-case scenario, not an average with respect to the whole time domain. As a result, integral inequalities could be avoided and the bound of τ will be obtained through linear equations. Clearly, the computational cost and root estimation of linear equations are generally better than exponential equations. We also give an example to show that a better bound on the observation duration τ could be obtained compared to that in [17] (a special case that time delays vanish).

2. Halanay-type inequality. We first give the notation to be used in this paper. Let $\mathbb{R}_+, \mathbb{N}_+$ be the family of all non-negative real numbers and natural numbers, respectively. Denote by $\mathbb{R}^d$ the d-dimensional vector space over the reals with Euclidean norm $|\cdot|$. If a and b are both real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Use $\mathbb{C}_h$ to represent the family of all continuous functions ξ from $[-h, 0]$ to $\mathbb{R}^d$ and designate the norm of its element ξ by $\|\xi\| = \sup_{-h\le\theta\le 0}|\xi(\theta)|$. Let $\mathbb{B}_a = \{\xi \in \mathbb{C}_h : \|\xi\| \le a\}$. For a vector or matrix M, M^{T} represents its transpose. If M is a matrix, denote its trace norm by $|M| = \sqrt{\mathrm{trace}(M^{\mathrm{T}}M)}$, and operator norm by $\|M\| = \max\{|Mx| : |x| = 1\}$. For a symmetric matrix M, let $\lambda_{\min}(M), \lambda_{\max}(M)$

be the smallest and largest eigenvalues, respectively. By $M > 0 (< 0)$, we mean M is positive (negative) definite. Also denote by I_d the d-dimensional identity matrix. For a set A, let 1_A be its indicator function with $1_A(a) = 1$ if $a \in A$, and 0 otherwise.

Let h_0 be a positive constant in this section. Next we provide the Halanay-type inequality, which forms a solid foundation when we study the boundedness and stability for delay equations subsequently.

THEOREM 2.1. *Assume that* $\Phi : [-h_0, \infty) \to \mathbb{R}_+$ *is a right-continuous function satisfying that* $\Phi(t) \leq C_\Phi$ *for* $t \in [-h_0, 0]$ *and*

$$D^+\Phi(t) \leq \lambda_1 - (\lambda_2 + \lambda_3(t))\Phi(t) + \lambda_3(t) \sup_{-h_0 \leq \theta \leq 0} \Phi(t+\theta), \quad \forall t \in \mathbb{R}_+. \tag{2.1}$$

Here $\lambda_1 \geq 0$, $\lambda_2 > 0$, *and* $\lambda_3(t)$ *is non-negative, bounded and right-continuous. Further assume that* $D^+\Phi(t)$ *is also a right-continuous function of time* t. *Then*

$$\Phi(t) \leq \frac{\lambda_1}{\lambda} + C_\Phi e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+, \tag{2.2}$$

where $\lambda > 0$ *is the unique root to equation* $\lambda - (\lambda_2 + \hat{\lambda}_3) + \hat{\lambda}_3 e^{\lambda h_0} = 0$, *in which* $\hat{\lambda}_3 = \sup_{t \in \mathbb{R}_+} \lambda_3(t)$, *and* D^+ *is the right-hand side Dini derivative.*

Proof. Since λ_2 is positive, λ is well defined. Furthermore, we have the following useful inequality

$$\lambda - (\lambda_2 + \lambda_3(t)) + \lambda_3(t) e^{\lambda h_0} \leq 0, \quad \forall t \in \mathbb{R}_+. \tag{2.3}$$

If not, we can find a $T \in \mathbb{R}_+$ so that $\lambda - (\lambda_2 + \lambda_3(T)) + \lambda_3(T) e^{\lambda h_0} > 0$. This implies that $(\lambda_2 + \hat{\lambda}_3) - \hat{\lambda}_3 e^{\lambda h_0} - (\lambda_2 + \lambda_3(T)) + \lambda_3(T) e^{\lambda h_0} > 0$. But this is impossible since the left-hand side equals $(\hat{\lambda}_3 - \lambda_3(T))\left(1 - e^{\lambda h_0}\right)$, which is non-positive. Thus we must have (2.3).

Next for each $t \in \mathbb{R}_+$, let

$$\Psi(t) = \sup_{-h_0 \leq \theta \leq 0} \left(e^{\lambda(t+\theta)} \Phi(t+\theta)\right), \qquad \psi(t) = \lambda_1 \int_0^t e^{\lambda s} \mathrm{d}s.$$

If we have

$$D^+(\Psi(t) - \psi(t)) \leq 0, \quad \forall t \in \mathbb{R}_+, \tag{2.4}$$

then we derive that

$$\Psi(t) - \psi(t) \leq \Psi(0) - \psi(0) = \sup_{-h_0 \leq \theta \leq 0} \Phi(\theta) \leq C_\Phi,$$

which implies that $e^{\lambda t}\Phi(t) \leq C_\Phi + \lambda_1 \int_0^t e^{\lambda s} \mathrm{d}s$. Dividing by $e^{\lambda t}$ on both sides gives the desired assertion (2.2).

The remaining work is to prove inequality (2.4). For any fixed $t \in \mathbb{R}_+$, either $\Psi(t) > e^{\lambda t}\Phi(t)$ or $\Psi(t) = e^{\lambda t}\Phi(t)$. In the former case, the right-continuity of $\Phi(\cdot)$ tells us that for any small $\Delta \in (0, h_0)$,

$$\Psi(t) \geq e^{\lambda s}\Phi(s), \quad s \in (t, t+\Delta].$$

While for $s \in [t + \Delta - h_0, t] \subset [t - h_0, t]$, it is naturally $e^{\lambda s}\Phi(s) \leq \Psi(t)$. Hence,

$$\Psi(t+\Delta) \leq \Psi(t) \leq \Psi(t) + \lambda_1 \int_t^{t+\Delta} e^{\lambda s} \mathrm{d}s,$$

which implies that $\Psi(t+\Delta)-\psi(t+\Delta)\le\Psi(t)-\psi(t)$, and so $D^+(\Psi(t)-\psi(t))\le 0$.

On the other hand, if we have $\Psi(t)=e^{\lambda t}\Phi(t)$, then

$$e^{\lambda(t+\theta)}\Phi(t+\theta)\le e^{\lambda t}\Phi(t),\quad \forall\theta\in[-h_0,0]. \tag{2.5}$$

Applying condition (2.1) and (2.3) yields that

$$\begin{aligned}D^+\Phi(t)+\lambda\Phi(t)\le&\lambda_1-(\lambda_2+\lambda_3(t))\Phi(t)+\lambda_3(t)\sup_{-h_0\le\theta\le 0}\Phi(t+\theta)+\lambda\Phi(t)\\ \le&\lambda_1+\left(\lambda_3(t)e^{\lambda h_0}-(\lambda_2+\lambda_3(t))+\lambda\right)\Phi(t)\\ \le&\lambda_1<\lambda_1+\varepsilon.\end{aligned}$$

Here ε is an arbitrary positive constant. By the right-continuity of $D^+\Phi(\cdot)$ and $\Phi(\cdot)$, for any small $\Delta\in(0,h_0)$, we also have

$$D^+\Phi(s)+\lambda\Phi(s)\le\lambda_1+\varepsilon,\quad\forall s\in(t,t+\Delta]. \tag{2.6}$$

Using the fundamental theorem of calculus and (2.6), for any $s\in(t,t+\Delta]$

$$\begin{aligned}e^{\lambda s}\Phi(s)\le&e^{\lambda t}\Phi(t)+\int_t^s e^{\lambda v}\left(D^+\Phi(v)+\lambda\Phi(v)\right)\mathrm{d}v\\ \le&e^{\lambda t}\Phi(t)+\lambda_1\int_t^{t+\Delta}e^{\lambda v}\mathrm{d}v+\varepsilon\int_t^{t+\Delta}e^{\lambda v}\mathrm{d}v.\end{aligned}$$

While for $s\in[t+\Delta-h_0,t]$, it is easy to see from (2.5) that

$$e^{\lambda s}\Phi(s)\le e^{\lambda t}\Phi(t)\le e^{\lambda t}\Phi(t)+\lambda_1\int_t^{t+\Delta}e^{\lambda v}\mathrm{d}v+\varepsilon\int_t^{t+\Delta}e^{\lambda v}\mathrm{d}v.$$

Combining these two inequalities, we obtain that

$$\Psi(t+\Delta)\le\Psi(t)+\lambda_1\int_t^{t+\Delta}e^{\lambda v}\mathrm{d}v+\varepsilon\int_t^{t+\Delta}e^{\lambda v}\mathrm{d}v,$$

and so $D^+(\Psi(t)-\psi(t))\le\varepsilon e^{\lambda t}$. Since t is fixed and $\varepsilon>0$ is arbitrary, we conclude that $D^+(\Psi(t)-\psi(t))\le 0$. This completes the proof of inequality (2.4). ∎

Remark 2.2. In [3, 32], the Halanay inequality is given by

$$D^+\Phi(t)\le\lambda_1-\hat\lambda_2(t)\Phi(t)+\lambda_3(t)\sup_{-h_0\le\theta\le 0}\Phi(t+\theta),\quad\forall t\in\mathbb{R}_+,$$

where $\hat\lambda_2(t),\lambda_3(t)$ are bounded continuous functions satisfying that $\inf_{t\in\mathbb{R}_+}(\hat\lambda_2(t)-\lambda_3(t))=\hat\sigma>0$. In this case, $\hat\lambda_2(t)\ge\lambda_3(t)+\hat\sigma$, which means our condition (2.1) holds with $\lambda_2=\hat\sigma$. Certainly, the key difference between our results with those in [3, 25, 32] is that $\Phi(t),\lambda_3(t)$ are all right-continuous. This change is actually non-trivial. Let us recall the proving ideas in [3, 25, 32]. The key step is to show the objective function $B(t):=e^{\lambda t}\Phi(t)-\psi(t)$ is smaller than C_Φ for all $t\in\mathbb{R}_+$. If this assertion is not true (namely, using the contradiction method), since $\Phi(t)$ is continuous, we are able to find $t_1>0$ such that (i) $B(t)<C_\Phi$ for all $0\le t<t_1$; (ii) $B(t_1)=C_\Phi$; (iii) $B(t_k)\ge C_\Phi$ for a decreasing sequence $\{t_k\}_{k\in\mathbb{N}_+}$ with $\lim_{k\to\infty}t_k=t_1$. Using (ii) and (iii) yields that $D^+B(t_1)\ge 0$, and using (i) and (ii) as well as condition (2.1) gives

that $D^+B(t_1) < 0$. But in this paper, we only have the right-continuity of $\Phi(t)$, which means (ii) will become $B(t_1) \geq C_\Phi$, and we could not derive $D^+B(t_1) \geq 0$ to identify the contradiction point. Thus in Theorem 2.1, we apply the supremum functional method, namely, construct the supremum functional $\sup_{-h_0\leq\theta\leq 0} B(t+\theta)$ to cover $B(t)$, analyze its Dini derivative, and obtain the decay trend. This idea is widely used in the stability study of delay equations such as the Razumikhin method [10, 23, 26]. But using this idea to prove the boundedness ($\lambda_1 > 0$) is a little technical. Finally, it should be pointed out that owing to the right-continuity of $\Phi(t)$ and the use of supremum functional method, we ask $D^+\Phi(t)$ to be right-continuous, which is crucial in the proof (see (2.6)). If $\Phi(t)$ is continuous and we use the contradiction method (e.g., [3, 32]), this requirement is not necessary.

When $\lambda_3(t)$ reduces to a constant, we get the following useful corollary.

COROLLARY 2.3. *Assume that $\Phi : [-h_0, \infty) \to \mathbb{R}_+$ is a right-continuous function satisfying that $\Phi(t) \leq M$ for $t \in [-h_0, 0]$ and*

$$D^+\Phi(t) \leq \lambda_1 - \lambda_2\Phi(t) + \lambda_3 \sup_{-h_0\leq\theta\leq 0} \Phi(t+\theta), \quad \forall t \in \mathbb{R}_+,$$

where constants $\lambda_1, \lambda_3 \geq 0$, $\lambda_2 > \lambda_3$. If $D^+\Phi(t)$ is right-continuous, then

$$\Phi(t) \leq \frac{\lambda_1}{\lambda} + Me^{-\lambda t}, \quad \forall t \in \mathbb{R}_+,$$

where λ is the unique root of the equation $\lambda - \lambda_2 + \lambda_3 e^{\lambda h_0} = 0$.

3. Boundedness.

Consider a d-dimensional hybrid SDDE

$$\text{(3.1)} \quad \mathrm{d}x(t) = f(x(t), x(t-\delta(t)), r(t))\mathrm{d}t + g(x(t), x(t-\delta(t)), r(t))\mathrm{d}W(t), \quad t \in \mathbb{R}_+,$$

defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ satisfying the usual conditions (that is, it is increasing, right-continuous and $\mathcal{F}_0$ contains all P-null sets). Here $W(t) = (W_1(t), \ldots, W_m(t))^{\mathrm{T}}$ is an m-dimensional Brownian motion, $r(t)$ is a right-continuous Markov chain independent of $W(t)$, taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, S\}$ with transition rate matrix $Q = (q_{ij})_{S\times S}$ given by

$$P\left(r(t+\Delta) = j | r(t) = i\right) = \begin{cases} 1 + q_{ij}\Delta + o(\Delta), & \text{if } i = j \\ q_{ij}\Delta + o(\Delta), & \text{if } i \neq j \end{cases}$$

as $\Delta \downarrow 0$. For hybrid SDDE (3.1), we need the initial data

$$\{x(\theta) : -h \leq \theta \leq 0\} = \xi \in \mathbb{C}_h, \quad r(0) = i_0 \in \mathbb{S}.$$

Assume that the Markov chain $r(t)$ is ergodic, and the delay function $\delta : \mathbb{R}_+ \to [0, h]$ is right-continuous; system coefficients $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous, and satisfy the following polynomial growth condition.

Assumption 3.1. There are positive constants K_0, K_1, K_2, and $p \geq 1$ such that for any $(x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$

$$\text{(3.2)} \quad |f(x, y, i)| \leq K_0 + K_1\left(|x|^p + |y|^p\right), \quad |g(x, y, i)|^2 \leq K_0 + K_2\left(|x|^{p+1} + |y|^{p+1}\right).$$

But this assumption could not guarantee the unique global solution of hybrid SDDE (3.1). Thus the following Khasminskii-type condition is often required (see, e.g., [23]).

Assumption 3.2. There is a positive constant α such that for any $(x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$

$$x^{\mathrm{T}} f(x, y, i) + \frac{2p-1}{2}|g(x, y, i)|^2 \leq \alpha \left(1 + |x|^2 + |y|^2\right). \tag{3.3}$$

Under these two assumptions, we can derive the existence and uniqueness of the global solution together with two useful moment properties.

THEOREM 3.3. *Let Assumptions 3.1 and 3.2 hold. Then for any initial data ξ and i_0, there exists a unique global solution $x^{\xi,i_0}(t)$ of hybrid SDDE (3.1), which satisfies that for all $T > 0$ and $R > 0$,*

$$\sup_{(\xi,i)\in\mathbb{B}_R\times\mathbb{S}} \sup_{0\leq t\leq T} E|x^{\xi,i_0}(t)|^{2p} \leq C_1(R, T), \tag{3.4}$$

$$\sup_{(\xi,i)\in\mathbb{B}_R\times\mathbb{S}} E\left(\sup_{0\leq t\leq T} |x^{\xi,i_0}(t)|^{p+1}\right) \leq C_2(R, T), \tag{3.5}$$

where $C_1(R, T)$ and $C_2(R, T)$ are positive constants depending on R and T.

To emphasize the initial data, we used $x^{\xi,i_0}(t)$ to denote the solution of hybrid SDDE (3.1) starting with (ξ, i_0). The existence of unique global solution as well as claim (3.4) is not difficult to show if we use the same analysis as in the proof of Theorem 2.6 in [15]. The other claim (3.5) could also be proved similar to Lemma 3.1 in [35]. But Theorem 3.3 only gives us a basic moment estimate of solution, it is not enough to obtain the moment boundedness. The additional assumption is needed.

Assumption 3.4. For each $i \in \mathbb{S}$, there are constants $\alpha_{1i}, \alpha_{3i} \geq 0$ and $\alpha_{2i} \in \mathbb{R}$ such that for any $(x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$

$$x^{\mathrm{T}} f(x, y, i) + \frac{1}{2}|g(x, y, i)|^2 \leq \alpha_{1i} + \alpha_{2i}|x|^2 + \alpha_{3i}|y|^2, \tag{3.6}$$

while for $A = -2\mathrm{diag}(\alpha_{21}, \cdots, \alpha_{2S}) - Q$ to be non-singular M-matrix and

$$2 \max_{i\in\mathbb{S}}(\eta_i\alpha_{3i}) < \frac{\check{\eta}}{\hat{\eta}}, \tag{3.7}$$

where $(\eta_1, \cdots, \eta_S)^{\mathrm{T}} = A^{-1}(1, \cdots, 1)^{\mathrm{T}}$ with $\hat{\eta} = \max_{i\in\mathbb{S}} \eta_i$ and $\check{\eta} = \min_{i\in\mathbb{S}} \eta_i$.

Since A is a non-singular M-matrix, all η_i are positive. Define the Lyapunov function $V(x, i) \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ by $V(x, i) = \eta_i|x|^2$, and the operator function $LV : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}$ by

$$LV(x, y, i) = 2\eta_i \left(x^{\mathrm{T}} f(x, y, i) + \frac{1}{2}|g(x, y, i)|^2\right) + \sum_{j\in\mathbb{S}} q_{ij}\eta_j|x|^2.$$

From condition (3.6), we derive that

$$\begin{aligned} LV(x, y, i) \leq& 2\eta_i \left(\alpha_{1i} + \alpha_{2i}|x|^2 + \alpha_{3i}|y|^2\right) + \sum_{j\in\mathbb{S}} q_{ij}\eta_j|x|^2 \\ =& 2\eta_i\alpha_{1i} + \left(2\eta_i\alpha_{2i} + \sum_{j\in\mathbb{S}} q_{ij}\eta_j\right)|x|^2 + 2\eta_i\alpha_{3i}|y|^2 \\ \leq& \hat{\alpha}_1 - |x|^2 + \hat{\alpha}_3|y|^2, \end{aligned} \tag{3.8}$$

where $\hat{\alpha}_1 = 2\max_{i\in\mathbb{S}}(\eta_i\alpha_{1i})$ and $\hat{\alpha}_3 = 2\max_{i\in\mathbb{S}}(\eta_i\alpha_{3i})$.

THEOREM 3.5. *Let Assumptions 3.1, 3.2, and 3.4 hold. Then for any initial data ξ and i_0,*

$$E|x^{\xi,i_0}(t)|^2 \le \frac{\hat{\alpha}_1}{\check{\eta}\alpha^*} + \frac{\hat{\eta}}{\check{\eta}}\|\xi\|^2, \quad \forall t \in \mathbb{R}_+, \tag{3.9}$$

where $\alpha^* = \frac{1}{\hat{\eta}} - \frac{\hat{\alpha}_3}{\check{\eta}} e^{\alpha^* h}$.

Proof. Fix ξ, i_0 and write $x(t) = x^{\xi,i_0}(t)$ for simplicity. Applying the generalized Itô formula to $V(x,i)$ gives that

$$EV(x(t),r(t)) = V_0 + \int_0^t ELV(x(s),x(s-\delta(s)),r(s))\mathrm{d}s, \tag{3.10}$$

where $V_0 = \eta_{i_0}|\xi(0)|^2$. Owing to the right-continuity of Markov chain $r(t)$ and delay function $\delta(t)$, $V(x(t),r(t))$ and $LV(x(t),x(t-\delta(t)),r(t))$ are right-continuous. For any $\Delta > 0$, we have

$$\sup_{t\le s\le t+\Delta} V(x(s),r(s)) \le \hat{\eta} \sup_{0\le s\le t+\Delta} |x(s)|^2 \le \hat{\eta}\left(1 + \sup_{0\le s\le t+\Delta} |x(s)|^{p+1}\right).$$

From condition (3.2), we also have

$$\begin{aligned}
&|LV(x(t),x(t-\delta(t)),r(t))| \\
\le& 2\hat{\eta}\left(|x(t)\|f(x(t),x(t-\delta(t)),r(t))| + \frac{1}{2}|g(x(t),x(t-\delta(t)),r(t))|^2\right) \\
&+ \sum_{j\in\mathbb{S}} |q_{r(t),j}|\eta_j|x(t)|^2 \\
\le& C\left(1 + |x(t)| + |x(t)|^{p+1} + |x(t-\delta(t))|^{p+1} + |x(t)|^2\right)
\end{aligned}$$

and as a result,

$$\sup_{t\le s\le t+\Delta} |LV(x(s),x(s-\delta(s)),r(s))| \le C\left(1 + \sup_{t-h\le s\le t+\Delta} |x(s)|^{p+1}\right),$$

where C is a genetic positive real constant independent of t and Δ. By (3.5), namely, $E\left(\sup_{0\le s\le t+\Delta} |x(s)|^{p+1}\right) < \infty$, the dominated convergence theorem yields that

$$\lim_{s\to t^+} EV(x(s),r(s)) = E\left(\lim_{s\to t^+} V(x(s),r(s))\right) = EV(x(t),r(t))$$

and

$$\begin{aligned}
\lim_{s\to t^+} ELV(x(s),x(s-\delta(s)),r(s)) =& E\left(\lim_{s\to t^+} LV(x(s),x(s-\delta(s)),r(s))\right) \\
=& ELV(x(t),x(t-\delta(t)),r(t)).
\end{aligned}$$

Thus $EV(x(t),r(t))$ and $ELV(x(t),x(t-\delta(t)),r(t))$ are right-continuous.

Next, let $\Phi(t) = EV(x(t),r(t))$ for $t \in \mathbb{R}_+$ and $\Phi(t) = \eta_{i_0}|\xi(t)|$ for $t \in [-h,0)$, then from (3.10), we have that $D^+\Phi(t) = ELV(x(t),x(t-\delta(t)),r(t))$, and also $\Phi(t)$ and $D^+\Phi(t)$ are right-continuous. From the estimate of LV in (3.8),

$$D^+\Phi(t) = ELV(x(t),x(t-\delta(t)),r(t))$$

$$\leq \hat{\alpha}_1 - \frac{1}{\hat{\eta}} E\left(\eta_{r(t)}|x(t)|^2\right) + \frac{\hat{\alpha}_3}{\check{\eta}} E\left(\eta_{r(t-\delta(t))}|x(t-\delta(t))|^2\right)$$

$$= \hat{\alpha}_1 - \frac{1}{\hat{\eta}} \Phi(t) + \frac{\hat{\alpha}_3}{\check{\eta}} \sup_{-h\leq\theta\leq 0} \Phi(t+\theta).$$

Since $\frac{1}{\hat{\eta}} > \frac{\hat{\alpha}_3}{\check{\eta}}$, which is guaranteed by condition (3.7), all conditions in Corollary 2.3 are satisfied. We conclude that

$$E|x(t)|^2 \leq \frac{1}{\check{\eta}} \Phi(t) \leq \frac{1}{\check{\eta}} \left(\frac{\hat{\alpha}_1}{\alpha^*} + \hat{\eta}\|\xi\|^2 e^{-\alpha^* t} \right). \tag{3.11}$$

The desired (3.9) follows immediately. The proof is hence complete. ∎

If we have $K_0 = 0$ in (3.2), then hybrid SDDE (3.1) admits the trivial solution. From (3.11), we have the following corollary of mean square exponential stability.

COROLLARY 3.6. *Let all the conditions in Theorem 3.5 hold. If $K_0 = 0$ and $\alpha_{1i} = 0$ for each $i \in \mathbb{S}$, then hybrid SDDE (3.1) is exponentially stable in mean square, that is, for any initial data ξ and i_0,*

$$\limsup_{t\to\infty} \frac{1}{t} \log E|x^{\xi,i_0}(t)|^2 \leq -\alpha^*. \tag{3.12}$$

4. Stability in distribution.

In the last section, we have seen the exponential stability in mean square when $K_0 = 0$ and all $\alpha_{1i} = 0$. But what happened if these conditions are not satisfied? This part studies the stability in distribution of hybrid SDDE (3.1). For this purpose, we introduce more notation. Let $r^{i_0}(t)$ denote the Markov chain starting from $r(0) = i_0$ and let $x^{\xi,i_0}(t)$ denote the solution of hybrid SDDE (3.1) with initial data ξ and i_0, and $x_t^{\xi,i_0} = \{x^{\xi,i_0}(t+\theta) : -h \leq \theta \leq 0\}$.

4.1. Time-homogeneity.

Although the joint process $(x_t^{\xi,i_0}, r^{i_0}(t))$ of past segment is indeed a Markov process for $t \in \mathbb{R}_+$, it might not be time-homogeneous in general. This is because hybrid SDDE (3.1) in fact, depends on t since our underlying delay function $\delta(t)$ is time-variable, not just a constant anymore. But the time-homogeneous Markov property is highly desired in the study of stability in distribution (see, e.g., [19, 38, 39]).

Therefore, in order to form time-homogeneous Markov processes, we need to assume that $\delta(t)$ is a periodic function with period $\mathcal{T}$. Without loss of generality, we may assume that $\mathcal{T} \geq h$; otherwise choose an integer n for $n\mathcal{T} \geq h$ and regard $\delta(t)$ as a periodic function with period $n\mathcal{T}$. We may further assume that $\mathcal{T} = h$; otherwise (i.e., $h < \mathcal{T}$) we can treat hybrid SDDE (3.1) as an SDDE with initial data $\xi_{\mathcal{T}} \in \mathbb{C}_{\mathcal{T}}$ by setting $\xi_{\mathcal{T}}(\theta) = \xi(\theta)$ for $\theta \in [-h, 0]$ and $\xi_{\mathcal{T}}(\theta) = \xi(-h)$ for $\theta \in [-\mathcal{T}, h)$ and resetting h to $\mathcal{T}$. From now on, we will assume that $\mathcal{T} = h$. With this standard setting, we could obtain the following time-homogeneous Markov process.

LEMMA 4.1. *The process $\{x_{kh}^{\xi,i_0}, r^{i_0}(kh)\}_{k\in\mathbb{N}_+}$ forms a time-homogeneous strong Markov process in $\mathbb{C}_h \times \mathbb{S}$. Its k-step transition probability is denoted as $p(k, \xi, i_0; \cdot \times \cdot)$, that is, for any Borel measurable set $\mathbb{U} \subset \mathbb{C}_h$ and $\mathbb{V} \subset \mathbb{S}$*

$$p(k, \xi, i_0; \mathbb{U} \times \mathbb{V}) = P((x_{kh}, r(kh)) \in \mathbb{U} \times \mathbb{V} \mid (x_0, r(0)) = (\xi, i_0)).$$

We refer the reader to Lemma 2.4 in [19] for the detailed proof of this lemma. Denote by $\mathbb{P}(\mathbb{C}_h)$ the family of all probability measures on $\mathbb{C}_h$. For $\mathcal{P}_1$ and $\mathcal{P}_2$ in

$\mathbb{P}(\mathbb{C}_h)$, define the Kantonorvich metric $d_{\mathbb{C}}$ by

$$d_{\mathbb{C}}(\mathcal{P}_1, \mathcal{P}_2) = \sup_{l \in \mathbb{L}} \left| \int_{\mathbb{C}_h} l(\zeta)\mathcal{P}_1(\mathrm{d}\zeta) - \int_{\mathbb{C}_h} l(\bar{\zeta})\mathcal{P}_2(\mathrm{d}\bar{\zeta}) \right|,$$

where $\mathbb{L} = \{l : \mathbb{C}_h \to \mathbb{R} : |l(\zeta) - l(\bar{\zeta})| \le \|\zeta - \bar{\zeta}\|$ and $|l(\cdot)| \le 1$ for any $\zeta, \bar{\zeta} \in \mathbb{C}_h\}$. Also let $\mathcal{P}(x_t^{\xi,i_0})$ be the probability measure on $\mathbb{C}_h$ generated by x_t^{ξ,i_0}. It is not hard to see that $\mathcal{P}(x_{kh}^{\xi,i_0}) = p(k, \xi, i_0; \cdot \times \mathbb{S})$ for each $k \in \mathbb{N}_+$.

Definition 4.2. *Hybrid SDDE (3.1) is said to be stable in distribution if there is a probability measure $\pi \in \mathbb{P}(\mathbb{C}_h)$ such that for any initial data ξ and i_0*

$$\lim_{k \to \infty} d_{\mathbb{C}}(\mathcal{P}(x_{kh}^{\xi,i_0}), \pi) = 0.$$

In the study of stability in distribution for hybrid SDDEs, several papers (see, e.g., [19, 39]) aimed to show the following two properties,

$$\sup_{k \in \mathbb{N}_+} E\|x_{kh}^{\xi,i_0}\|^2 < \infty \quad \text{and} \quad \lim_{k \to \infty} E\|x_{kh}^{\xi,i_0} - x_{kh}^{\bar{\xi},i_0}\|^2 = 0.$$

The former is the moment boundedness of the segment process x_{kh}^{ξ,i_0}, and the latter is so-called the moment attractivity of x_{kh} starting from different initial values ξ and $\bar{\xi}$. But they are a little strong sometimes since we might impose some additional conditions to obtain these two moment properties. In fact, according to the theory in [5, 30, 38], they are only needed in the probability version.

Definition 4.3. *Hybrid SDDE (3.1) is said to have property (P1) if for any $(\xi, i_0) \in \mathbb{C}_h \times \mathbb{S}$ and any $\varepsilon > 0$, there is a positive number $R_1 = R_1(\varepsilon, \xi)$ such that*

$$P(\|x_{kh}^{\xi,i_0}\| \le R_1) \ge 1 - \varepsilon, \quad \forall k \in \mathbb{N}_+. \tag{4.1}$$

Hybrid SDDE (3.1) is said to have property (P2) if for any $\varepsilon > 0$ and any $R > 0$, there is a $k_0 = k_0(\varepsilon, R)$ such that

$$P(\|x_{kh}^{\xi,i_0} - x_{kh}^{\bar{\xi},i_0}\| \le \varepsilon) \ge 1 - \varepsilon, \quad \forall k > k_0, \tag{4.2}$$

whenever $\xi, \bar{\xi} \in \mathbb{B}_R$ and $i_0 \in \mathbb{S}$.

Property (P1) is referred to as the boundedness in probability of segment processes, which could be derived in the following lemma.

Lemma 4.4. *Under Assumptions 3.1, 3.2 and 3.4, hybrid SDDE (3.1) has property (P1).*

Proof. Let ξ, i_0, ε be fixed, and write $x(t) = x^{\xi,i_0}(t)$, $x_t = x_t^{\xi,i_0}$. For any $t \in \mathbb{R}_+$ and any positive constant H, define a stopping time $\sigma_t^H = \inf\{s \ge t : |x(s)| > H\}$. Applying the Itô formula to $e^t|x|^2$ and using condition (3.6) gives that

$$\begin{aligned} &E\left(e^{(t+h)\wedge\sigma_t^H}|x((t+h)\wedge\sigma_t^H)|^2\right) \\ \le& e^t E|x(t)|^2 + E\int_t^{(t+h)\wedge\sigma_t^H} Ce^s\left(1 + |x(s)|^2 + |x(s-\delta(s))|^2\right)\mathrm{d}s \\ \le& e^t E|x(t)|^2 + C\int_t^{t+h} e^s\left(1 + E|x(s)|^2 + E|x(s-\delta(s))|^2\right)\mathrm{d}s, \end{aligned}$$

where C is a genetic positive real constant independent of t, H, ξ, and i_0. By Theorem 3.5, there exists a positive constant $R_2 = R_2(\xi)$ such that $E|x(v)|^2 \le R_2$ for any $v \ge 0$. Consequently,

$$e^t E|x((t+h)\wedge\sigma_t^H)|^2 \le E\left(e^{(t+h)\wedge\sigma_t^H}|x((t+h)\wedge\sigma_t^H)|^2\right) \le R_2 e^t + C(1+2R_2)e^{t+h}.$$

Letting $R_3 = \sqrt{\frac{1}{\varepsilon}(R_2 + C(1+2R_2)e^h)}$, we have $P(\sigma_t^{R_3} \le t+h) \le \varepsilon$, which implies

$$P\left(\sup_{t\le s\le t+h}|x(s)| \le R_3\right) \ge 1-\varepsilon, \quad \forall t \in \mathbb{R}_+.$$

Taking $R_1 = R_3 + \|\xi\|$, which depends on ε and ξ, we have

$$P(\|x_t\| \le R_1) = P\left(\sup_{t-h\le s\le t}|x(s)| \le R_1\right) \ge 1-\varepsilon, \quad \forall t \in \mathbb{R}_+.$$

Inequality (4.1) follows naturally by taking $t = kh$. The proof is hence complete. ∎

4.2. Attractivity. For property (P2), we need to consider the difference equation of two solutions with initial data (ξ, i_0) and $(\bar{\xi}, i_0)$

$$\begin{aligned}
&\mathrm{d}(x^{\xi,i_0}(t) - x^{\bar{\xi},i_0}(t))\\
&= \left(f(x^{\xi,i_0}(t), x^{\xi,i_0}(t-\delta(t)), r^{i_0}(t)) - f(x^{\bar{\xi},i_0}(t), x^{\bar{\xi},i_0}(t-\delta(t)), r^{i_0}(t))\right)\mathrm{d}t\\
&+ \left(g(x^{\xi,i_0}(t), x^{\xi,i_0}(t-\delta(t)), r^{i_0}(t)) - g(x^{\bar{\xi},i_0}(t), x^{\bar{\xi},i_0}(t-\delta(t)), r^{i_0}(t))\right)\mathrm{d}W(t).
\end{aligned} \tag{4.3}$$

And the following attractivity condition is needed.

Assumption 4.5. Assume that for each $i \in \mathbb{S}$, there are constants $\beta_{2i} \in \mathbb{R}$ and $\beta_{3i} \ge 0$ such that for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d, i \in \mathbb{S}$

$$\begin{aligned}
&(x-\bar{x})^{\mathrm{T}}(f(x,y,i) - f(\bar{x},\bar{y},i)) + \frac{1}{2}|g(x,y,i) - g(\bar{x},\bar{y},i)|^2\\
&\le \beta_{2i}|x-\bar{x}|^2 + \beta_{3i}|y-\bar{y}|^2,
\end{aligned} \tag{4.4}$$

while for $B = -2\mathrm{diag}(\beta_{21}, \cdots, \beta_{2S}) - Q$ to be a non-singular M-matrix and

$$2\max_{i\in\mathbb{S}}(\gamma_i\beta_{3i}) < \frac{\check{\gamma}}{\hat{\gamma}}, \tag{4.5}$$

where $(\gamma_1, \ldots, \gamma_S)^{\mathrm{T}} = B^{-1}(1, \ldots, 1)^{\mathrm{T}}$ with $\hat{\gamma} = \max_{i\in\mathbb{S}}\gamma_i$ and $\check{\gamma} = \min_{i\in\mathbb{S}}\gamma_i$.

Define the Lyapunov function $U(x,i) \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ by $U(x,i) = \gamma_i|x|^2$, and the operator function $\mathcal{L}U : \mathbb{R}^{4d} \times \mathbb{S} \to \mathbb{R}$ corresponding to equation (4.3) by

$$\begin{aligned}
\mathcal{L}U(x,\bar{x},y,\bar{y},i) =& 2\gamma_i\left((x-\bar{x})^{\mathrm{T}}(f(x,y,i) - f(\bar{x},\bar{y},i)) + \frac{1}{2}|g(x,y,i) - g(\bar{x},\bar{y},i)|^2\right)\\
&+ \sum_{j\in\mathbb{S}} q_{ij}\gamma_j|x-\bar{x}|^2.
\end{aligned}$$

Using (4.4) and letting $\hat{\beta}_3 = 2\max_{i\in\mathbb{S}}(\gamma_i\beta_{3i})$, it is easy to derive that

$$\mathcal{L}U(x,\bar{x},y,\bar{y},i) \le -|x-\bar{x}|^2 + \hat{\beta}_3|y-\bar{y}|^2. \tag{4.6}$$

LEMMA 4.6. *Let Assumptions 3.1, 3.2, 3.4, and 4.5 hold. Then for any positive number R, and $\xi,\bar\xi\in\mathbb{B}_R$, $i_0\in\mathbb{S}$,*

$$E|x^{\xi,i_0}(t)-x^{\bar\xi,i_0}(t)|^2\le\frac{4\hat\gamma R^2}{\check\gamma}e^{-\gamma^* t}, \tag{4.7}$$

where γ^* is the root to equation $\gamma^*=\frac{1}{\check\gamma}-\frac{\hat\beta_3}{\check\gamma}e^{\gamma^* h}$.

Proof. Fix $\xi,\bar\xi\in\mathbb{B}_R$, $i_0\in\mathbb{S}$ and write $x(t)=x^{\xi,i_0}(t)$, $\bar x(t)=x^{\bar\xi,i_0}(t)$ simply. Applying the generalized Itô formula to equation (4.3), we have

$$EU(x(t)-\bar x(t),r(t))=U_0+\int_0^t E\mathcal{L}U(x(s),x(s-\delta(s)),\bar x(s),\bar x(s-\delta(s)),r(s))\mathrm{d}s,$$

where $U_0=\gamma_{i_0}|\xi(0)-\bar\xi(0)|^2\le\hat\gamma\|\xi-\bar\xi\|^2$. By the definition of U and $\mathcal{L}U$, we compute

$$U(x-\bar x,i)\le\hat\gamma|x-\bar x|^2\le C\left(1+|x|^{p+1}+|\bar x|^{p+1}\right)$$

and

$$\begin{aligned}|\mathcal{L}U(x,y,\bar x,\bar y,i)|\le&C\Big(|x|+|\bar x|)(|f(x,y,i)|+|f(\bar x,\bar y,i)|)\\&+|g(x,y,i)|^2+|g(\bar x,\bar y,i)|^2+|x|^2+|\bar x|^2\Big)\\ \le&C\left(1+|x|^{p+1}+|y|^{p+1}+|\bar x|^{p+1}+|\bar y|^{p+1}\right).\end{aligned}$$

For any $\Delta>0$, $\sup_{t\le s\le t+\Delta}U(x(s)-\bar x(s),r(s))$ and $\sup_{t\le s\le t+\Delta}\mathcal{L}U(x(s),x(s-\delta(s)),\bar x(s),\bar x(s-\delta(s)),r(s))$ are bounded by

$$C\left(1+\sup_{0\le s\le t+\Delta}|x(s)|^{p+1}+\sup_{0\le s\le t+\Delta}|\bar x(s)|^{p+1}\right).$$

From Theorem 3.3, the expectation of this item is finite, so that by the dominated convergence theorem again, $EU(x(t)-\bar x(t),r(t))$ and $E\mathcal{L}U(x(t),x(t-\delta(t)),\bar x(t),\bar x(t-\delta(t)),r(t))$ are right-continuous. Choose $\Phi(t)$ by $EU(x(t)-\bar x(t),r(t))$, then $D^+\Phi(t)=E\mathcal{L}U(x(t),x(t-\delta(t)),\bar x(t),\bar x(t-\delta(t)),r(t))$. The estimate of $\mathcal{L}U$ in (4.6) further gives that

$$\begin{aligned}D^+\Phi(t)\le&-|x(t)-\bar x(t)|^2+\hat\beta_3|x(t-\delta(t))-\bar x(t-\delta(t))|^2\\ \le&-\frac{1}{\check\gamma}\Phi(t)+\frac{\hat\beta_3}{\check\gamma}\sup_{-h\le\theta\le0}\Phi(t+\theta).\end{aligned}$$

By condition (4.5), $\frac{1}{\check\gamma}>\frac{\hat\beta_3}{\check\gamma}$. We could hence derive from Corollary 2.3 that

$$E|x(t)-\bar x(t)|^2\le\frac{1}{\check\gamma}\Phi(t)\le\frac{\hat\gamma}{\check\gamma}\|\xi-\bar\xi\|^2e^{-\gamma^* t}\le\frac{4\hat\gamma R^2}{\check\gamma}e^{-\gamma^* t},$$

The proof is concluded. ∎

We need to extend the convergence of two solutions from point distance into segment distance. If we follow the proof of Lemma 3.2 in [38], we could derive that $\lim_{k\to\infty}E\|x^{\xi,i_0}_{kh}-x^{\bar\xi,i_0}_{kh}\|^2=0$, though we would impose an additional condition on f and g, namely,

$$(x-\bar x)^{\mathrm{T}}(f(x,y,i)-f(\bar x,\bar y,i))+17|g(x,y,i)-g(\bar x,\bar y,i)|^2\le\beta_{2i}|x-\bar x|^2+\beta_{3i}|y-\bar y|^2.$$

To avoid this, we need to prove a weaker result as follows.

LEMMA 4.7. *Under Assumptions 3.1, 3.2, 3.4, and 4.5, hybrid SDDE (3.1) has property (P2).*

Proof. Let $R > 0$ and $\varepsilon > 0$ be fixed. For any given $\xi, \bar{\xi} \in \mathbb{B}_R, i_0 \in \mathbb{S}$, write $x(t) = x^{\xi,i_0}(t)$, $x_t = x_t^{\xi,i_0}$, $\bar{x}(t) = x^{\bar{\xi},i_0}(t)$, $\bar{x}_t = x_t^{\bar{\xi},i_0}$. By Lemma 4.6, for any $\varepsilon_0 > 0$, there is a $T_0 = T_0(\varepsilon_0, R)$ such that

$$E|x(t) - \bar{x}(t)|^2 \le \varepsilon_0, \quad \forall t \ge T_0. \tag{4.8}$$

For any $t \ge T_0 + h$, define a stopping time $\bar{\sigma}_t^\varepsilon = \inf\{s \ge t : |x(s) - \bar{x}(s)| > \varepsilon\}$. Applying the Itô formula to $e^t|x|^2$ and using condition (4.4) shows that

$$\begin{aligned}
&E\left(e^{(t+h)\wedge\bar{\sigma}_t^\varepsilon}|x((t+h)\wedge\bar{\sigma}_t^\varepsilon) - \bar{x}((t+h)\wedge\bar{\sigma}_t^\varepsilon)|^2\right) - e^t E|x(t) - \bar{x}(t)|^2 \\
\le& E\int_t^{(t+h)\wedge\bar{\sigma}_t^\varepsilon} Ce^s\left(|x(s) - \bar{x}(s)|^2 + |x(s-\delta(s)) - \bar{x}(s-\delta(s))|^2\right) \mathrm{d}s \\
\le& C\int_t^{t+h} e^s\left(E|x(s) - \bar{x}(s)|^2 + E|x(s-\delta(s)) - \bar{x}(s-\delta(s))|^2\right) \mathrm{d}s,
\end{aligned}$$

where C is a genetic positive number independent of t, ε_0, ε. By (4.8),

$$e^t E|x((t+h)\wedge\bar{\sigma}_t^\varepsilon) - \bar{x}((t+h)\wedge\bar{\sigma}_t^\varepsilon)|^2 \le \varepsilon_0 e^t + 2C\varepsilon_0 e^{t+h}.$$

Taking $\varepsilon_0 = \frac{\varepsilon^3}{1+2Ce^h}$, we have $P(\bar{\sigma}_t^\varepsilon \le t+h) \le \frac{1}{\varepsilon^2}(\varepsilon_0 + 2C\varepsilon_0 e^h) \le \varepsilon$. Thus

$$P\left(\sup_{t\le s\le t+h}|x(s) - \bar{x}(s)| \le \varepsilon\right) \ge 1 - \varepsilon, \quad \forall t \ge T_0 + h.$$

Let k_0 be an integer such that $k_0 h \ge T_0 + 2h$, which depends on ε and R. For any $k > k_0$, we have

$$P(\|x_{kh} - \bar{x}_{kh}\| \le \varepsilon) = P\left(\sup_{kh-h\le s\le kh}|x(s) - \bar{x}(s)| \le \varepsilon\right) \ge 1 - \varepsilon.$$

In other words, property (P2) holds. This completes the proof. ∎

4.3. Stability results. After the preparation of Lemmas 4.4 and 4.7, we can state the results of stability in distribution. Here, we could see the significance of identifying time-homogeneous Markov processes (e.g., (4.12)).

THEOREM 4.8. *Let Assumptions 3.1, 3.2, 3.4, and 4.5 hold. Hybrid SDDE (3.1) is stable in distribution.*

Proof. Step 1. We claim that for any given $R > 0$,

$$\lim_{k\to\infty} d_{\mathbb{C}}(\mathcal{P}(x_{kh}^{\xi,i_0}), \mathcal{P}(x_{kh}^{\bar{\xi},\bar{i}_0})) = 0 \tag{4.9}$$

whenever $\xi, \bar{\xi} \in \mathbb{B}_R$, $i_0, \bar{i}_0 \in \mathbb{S}$. In fact, since $\{r(kh)\}_{k\in\mathbb{N}_+}$ is an ergodic Markov chain, the stopping time $\kappa_{i_0,\bar{i}_0} = \inf\{kh : r^{i_0}(kh) = r^{\bar{i}_0}(kh), k \in \mathbb{N}_+\}$ is finite almost surely. Then for any $\varepsilon > 0$, there is a positive $T_1 = T_1(\varepsilon)$ such that $P(\kappa_{i_0,\bar{i}_0} \le T_1) > 1 - \frac{\varepsilon}{6}$ for any $i_0, \bar{i}_0 \in \mathbb{S}$. From (3.5), we have

$$\sup_{(\xi,i)\in\mathbb{B}_R\times\mathbb{S}} E\left(\sup_{0\le t\le T_1}|x^{\xi,i_0}(t)|^{p+1}\right) < \infty,$$

and so there is a large number $R_4 = R_4(\varepsilon, R)$ such that $P(\Omega_{\xi,i_0}) > 1 - \frac{\varepsilon}{12}$ for any $(\xi, i_0) \in \mathbb{B}_R \times \mathbb{S}$, where $\Omega_{\xi,i_0} = \{\omega : \sup_{0\le t\le T_1} |x^{\xi,i_0}(t,\omega)| \le R_4\}$. Fixing $\xi, \bar\xi \in \mathbb{B}_R$ and $i_0, \bar i_0 \in \mathbb{S}$, we see that for any $l \in \mathbb{L}$ and $k \in \mathbb{N}_+$ with $k \ge k_{T_1} := [T_1/h] + 1$,

$$\begin{aligned}
\Big|El(x^{\xi,i_0}_{kh}) - El(x^{\bar\xi,\bar i_0}_{kh})\Big| \le& 2P(\kappa_{i_0,\bar i_0} > T_1) + E\left(1_{\{\kappa_{i_0,\bar i_0}\le T_1\}}|l(x^{\xi,i_0}_{kh}) - l(x^{\bar\xi,\bar i_0}_{kh})|\right)\\
\le& \frac{\varepsilon}{3} + E\left(1_{\{\kappa_{i_0,\bar i_0}\le T_1\}} E\left(|l(x^{\xi,i_0}_{kh}) - l(x^{\bar\xi,\bar i_0}_{kh})|\Big|\mathcal{F}_{\kappa_{i_0,\bar i_0}}\right)\right)\\
\le& \frac{2\varepsilon}{3} + E\left(1_{\Omega_0} E\left(2\wedge \|x^{u,j}_{kh-\kappa_{i_0,\bar i_0}} - x^{v,j}_{kh-\kappa_{i_0,\bar i_0}}\|\right)\right), \qquad (4.10)
\end{aligned}$$

where $\Omega_0 = \{\kappa_{i_0,\bar i_0} \le T_1\} \cap \Omega_{\xi,i_0} \cap \Omega_{\bar\xi,\bar i_0}$, and $u = x^{\xi,i_0}_{\kappa_{i_0,\bar i_0}}$, $v = x^{\bar\xi,\bar i_0}_{\kappa_{i_0,\bar i_0}}$, $j = r^{i_0}(\kappa_{i_0,\bar i_0}) = r^{\bar i_0}(\kappa_{i_0,\bar i_0})$. Since for $\omega \in \Omega_0$, $\|u\|, \|v\| \le R_4$, we derive from Lemma 4.7 (namely property (P2)) that there is a $k_P \in \mathbb{N}_+$ such that for all $k > k_P + k_{T_1}$,

$$E\left(2\wedge \|x^{u,j}_{kh-\kappa_{i_0,\bar i_0}} - x^{v,j}_{kh-\kappa_{i_0,\bar i_0}}\|\right) \le \frac{\varepsilon}{3}.$$

Substituting this into (4.10) gives that $\Big|El(x^{\xi,i_0}_{kh}) - El(x^{\bar\xi,\bar i_0}_{kh})\Big| \le \varepsilon$ for any $k > k_P + k_{T_1}$. Since l is arbitrary, and $k_P + k_{T_1}$ only depends on ε, R, we must have claim (4.9).

Step 2. For any fixed $(\xi, i_0) \in \mathbb{C}_h \times \mathbb{S}$, $\{\mathcal{P}(x^{\xi,i_0}_{kh})\}_{k\in\mathbb{N}_+}$ is a Cauchy sequence with $d_{\mathbb{C}}$ in $\mathbb{P}(\mathbb{C}_h)$. This is equivalent to show that for any $\varepsilon > 0$, there is a $k_C \in \mathbb{N}_+$ such that

$$\sup_{l\in\mathbb{L}} \Big|El(x^{\xi,i_0}_{(k+k_1)h}) - El(x^{\xi,i_0}_{kh})\Big| \le \varepsilon, \quad \forall k > k_C, k_1 \ge 1. \qquad (4.11)$$

For each $l \in \mathbb{L}$, define a function ϕ_l with respect to l, that $\phi_l(u, i) = \sum_{j\in\mathbb{S}} l(u)1_{\{j\}}(i)$ for each $(u, i) \in \mathbb{C}_h \times \mathbb{S}$. It is easy to see that for any $i \in \mathbb{S}$, $\phi_l(\cdot, i) = l(\cdot)$. Then compute for any $k, k_1 \ge 1$,

$$\begin{aligned}
El(x^{\xi,i_0}_{(k+k_1)h}) =& E\left(E\left(\phi_l(x^{\xi,i_0}_{(k+k_1)h}, r^{i_0}((k+k_1)h))\Big|\mathcal{F}_{k_1 h}\right)\right)\\
=& \int_{\mathbb{C}_h\times\mathbb{S}} E\left(\phi_l(x^{\xi,i_0}_{(k+k_1)h}, r^{i_0}((k+k_1)h))\Big|(x^{\xi,i_0}_{k_1 h}, r^{i_0}(k_1 h)) = (u, j)\right)\\
& \times p(k, \xi, i_0; \mathrm{d}u \times \{j\})\\
=& \int_{\mathbb{C}_h\times\mathbb{S}} E\phi_l(x^{u,j}_{kh}, r^j(kh)) p(k_1, \xi, i_0; \mathrm{d}u \times \{j\})\\
=& \int_{\mathbb{C}_h\times\mathbb{S}} El(x^{u,j}_{kh}) p(k_1, \xi, i_0; \mathrm{d}u \times \{j\}). \qquad (4.12)
\end{aligned}$$

Here we have used the Markov property of the process $\{x^{\xi,i_0}_{kh}, r^{i_0}(kh)\}$ in the second equality, and its time-homogeneous property in the third equality. Since ξ is fixed, we can find a number $R_5 > 0$ such that $\|\xi\| \le R_5$. By Lemma 4.4 (namely property (P1)), for any $\varepsilon > 0$, there is a number $R_1 > 0$ such that $P\left(\|x^{\xi,i_0}_{k_1 h}\| \le R_1\right) \ge 1 - \frac{\varepsilon}{4}$ for all $k_1 \ge 1$. Then letting $R_6 = R_5 \vee R_1$ and using (4.12), we have

$$\begin{aligned}
\Big|El(x^{\xi,i_0}_{(k+k_1)h}) - El(x^{\xi,i_0}_{kh})\Big| \le& \int_{\mathbb{C}_h\times\mathbb{S}} \Big|El(x^{u,j}_{kh}) - El(x^{\xi,i_0}_{kh})\Big| p(k_1, \xi, i_0; \mathrm{d}u \times \{j\})\\
\le& \frac{2\varepsilon}{4} + \int_{\mathbb{B}_{R_6}\times\mathbb{S}} d_{\mathbb{C}}(\mathcal{P}(x^{u,j}_{kh}), \mathcal{P}(x^{\xi,i_0}_{kh})) p(k_1, \xi, i_0; \mathrm{d}u \times \{j\}).
\end{aligned}$$

From (4.9), we can find $k_C \in \mathbb{N}_+$ such that $d_{\mathbb{C}}(\mathcal{P}(x^{u,j}_{kh}), \mathcal{P}(x^{\xi,i_0}_{kh})) \leq \frac{\varepsilon}{2}$ for any $k > K_C$ whenever $(u,j) \in \mathbb{B}_{R_6} \times \mathbb{S}$. Thus $\left| El(x^{\xi,i_0}_{(k+k_1)h}) - El(x^{\xi,i_0}_{kh}) \right| \leq \varepsilon$. Since l is arbitrary, (4.11) follows naturally.

Step 3. There is a unique $\pi \in \mathbb{P}(\mathbb{C}_h)$ such that $\lim_{k\to\infty} d_{\mathbb{C}}(\mathcal{P}(x^{0,1}_{kh}), \pi) = 0$. This together with (4.9) implies that for any $(\xi, i_0) \in \mathbb{C}_h \times \mathbb{S}$,

$$\lim_{k\to\infty} d_{\mathbb{C}}(\mathcal{P}(x^{\xi,i_0}_{kh}), \pi) \leq \lim_{k\to\infty} d_{\mathbb{C}}(\mathcal{P}(x^{\xi,i_0}_{kh}), \mathcal{P}(x^{0,1}_{kh})) + \lim_{k\to\infty} d_{\mathbb{C}}(\mathcal{P}(x^{0,1}_{kh}), \pi) = 0.$$

This completes the proof. ◻

As discussed in [19], since it is difficult to numerically approximate the invariant measure π in the huge space $\mathbb{P}(\mathbb{C}_h)$, it is necessary to consider the long-time behavior of $x^{\xi,i_0}(t)$, where the measure space becomes the smaller $\mathbb{P}(\mathbb{R}^d)$ (the family of probability measures on $\mathbb{R}^d$). Let $\hat{\mathcal{P}}(x^{\xi,i_0}(t))$ be the probability measure on $\mathbb{R}^d$ generated by $x^{\xi,i_0}(t)$. Define the Kantonorvich metric in $\mathbb{P}(\mathbb{R}^d)$

$$d_{\mathbb{R}}(\hat{\mathcal{P}}_1, \hat{\mathcal{P}}_2) = \sup_{l^0 \in \mathbb{L}^0} \left| \int_{\mathbb{R}^d} l^0(x) \hat{\mathcal{P}}_1(\mathrm{d}x) - \int_{\mathbb{R}^d} l^0(\bar{x}) \hat{\mathcal{P}}_2(\mathrm{d}\bar{x}) \right|,$$

where $\mathbb{L}^0 = \{l^0 : \mathbb{R}^d \to \mathbb{R} : |l^0(x) - l^0(\bar{x})| \leq |x - \bar{x}|$ and $|l^0(\cdot)| \leq 1$ for any $x, \bar{x} \in \mathbb{R}^d\}$. Then we have the following useful corollary in practice.

COROLLARY 4.9. *Let all the conditions in Theorem 4.8 hold. Then for any fixed $h^0 \in [0,h)$, there is a unique $\pi_{h^0} \in \mathbb{P}(\mathbb{R}^d)$ such that for any initial data ξ and i_0*

$$\lim_{k\to\infty} d_{\mathbb{R}}(\hat{\mathcal{P}}(x^{\xi,i_0}(h^0 + kh)), \pi_{h^0}) = 0. \tag{4.13}$$

Proof. Let h^0 be fixed and define a mapping Π from $\mathbb{C}_h$ to $\mathbb{R}^d$ by $\Pi(\hat{\xi}) = \hat{\xi}(h^0 - h)$ for any $\hat{\xi} \in \mathbb{C}_h$. It is easy to see that Π is continuous, and $l^0(\Pi) \in \mathbb{L}$ for any $l^0 \in \mathbb{L}^0$. For any $\xi, \bar{\xi} \in \mathbb{C}_h$, $i_0, \bar{i}_0 \in \mathbb{S}$, $k \in \mathbb{N}_+$, we have

$$\begin{aligned}
&d_{\mathbb{R}}(\hat{\mathcal{P}}(x^{\xi,i_0}(h^0 + kh)), \hat{\mathcal{P}}(x^{\bar{\xi},\bar{i}_0}(h^0 + kh)))\\
=& \sup_{l^0 \in \mathbb{L}^0} |El^0(x^{\xi,i_0}(h^0 + kh)) - El^0(x^{\bar{\xi},\bar{i}_0}(h^0 + kh))|\\
=& \sup_{l^0 \in \mathbb{L}^0} |El^0(\Pi(x^{\xi,i_0}_{kh})) - El^0(\Pi(x^{\bar{\xi},\bar{i}_0}_{kh}))|\\
\leq& \sup_{l^0 \in \mathbb{L}^0} \sup_{l \in \mathbb{L}} |El(x^{\xi,i_0}_{kh}) - El(x^{\bar{\xi},\bar{i}_0}_{kh})|\\
=& d_{\mathbb{C}}(\mathcal{P}(x^{\xi,i_0}_{kh}), \mathcal{P}(x^{\bar{\xi},\bar{i}_0}_{kh})).
\end{aligned}$$

Thus we derive similar results as (4.9) and (4.11). The required assertion (4.13) then follows naturally. ◻

5. Discrete-time feedback control. When the given hybrid SDDE (3.1) is not stable in distribution, we need to design a feedback control based on discrete-time observations to achieve the stabilization. The controlled SDDE is given by

$$\begin{aligned}
\mathrm{d}x(t) =& \Big(f(x(t), x(t-\delta(t)), r(t)) + D_{r(t)} x([t/\tau]\tau) \Big) \mathrm{d}t\\
&+ g(x(t), x(t-\delta(t)), r(t)) \mathrm{d}W(t)
\end{aligned} \tag{5.1}$$

with initial data $\xi \in \mathbb{C}_h$, $i_0 \in \mathbb{S}$. If we let $H(t) = t - [t/\tau]\tau$, the control term can be written as $D_{r(t)}x(t-H(t))$ so we see that the controlled SDDE (5.1) is a delay system with two delays, and the Halany-type results are well-applicable.

First, due to the technical need of studying discrete-time feedback control, the following assumption on the original unstable hybrid SDDE is imposed.

Assumption 5.1. Assume that there are positive constants L_1, L_2, L_3, and L_4 such that for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ and $i \in \mathbb{S}$

$$\begin{aligned} |f(x,y,i) - f(\bar{x},\bar{y},i)| \le & L_1|x-\bar{x}| + L_2|y-\bar{y}|, \\ |g(x,y,i) - g(\bar{x},\bar{y},i)| \le & L_3|x-\bar{x}| + L_4|y-\bar{y}|. \end{aligned}$$

In what follows, we common on why we impose Assumption 5.1.

Remark 5.2. Assumption 5.1 is the global Lipschitz condition, which could guarantee the controlled SDDE (5.1) admits a unique global solution, satisfying that $E\left(\sup_{0\le t\le T} |x^{\xi,i_0}(t)^2|\right) < \infty$ for all $T > 0$, since f and g also meet the linear growth condition from this assumption, namely, for any $(x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$

$$\begin{aligned} |f(x,y,i)| \le & |f(x,y,i) - f(0,0,i)| + |f(0,0,i)| \le K_3 + L_1|x| + L_2|y|, \\ |g(x,y,i)| \le & |g(x,y,i) - g(0,0,i)| + |g(0,0,i)| \le K_4 + L_3|x| + L_4|y|, \end{aligned}$$

where $K_3 = \max_{i\in\mathbb{S}} |f(0,0,i)|$ and $K_4 = \max_{i\in\mathbb{S}} |g(0,0,i)|$. But if just for the global solution, the local Lipschitz condition with Assumption 3.2 is enough. Why do we need a stronger one? This is because the study of discrete-time feedback control is a kind of delay-dependent property. A significant difference from the delay-independent analysis in Sections 3 and 4, is that the delay-dependence analysis in this section is to estimate $|x^{\xi,i_0}(t) - x^{\xi,i_0}(t-H(t))|$. If τ is sufficiently small, then this difference goes to zero, as a result, we do not need to strengthen the non-delay part to suppress the delay effect like conditions (3.7) or (4.5). However, in the sense of stability in distribution, the case becomes more complicated, since we would compute $|(x^{\xi,i_0}(t) - x^{\bar{\xi},\bar{i}_0}(t)) - (x^{\xi,i_0}(t-H(t)) - x^{\bar{\xi},\bar{i}_0}(t-H(t)))|$. This force us to impose Assumption 5.1, whose details could be found in the proof of Lemma 5.7. It is still a challenge problem how to estimate this difference without the global Lipschitz condition.

To make the design of feedback control simpler, we consider the linear form of control function $D_i x$. To design the matrices D_i and the observation duration τ is a non-trivial issue. To design D_i, we use the following rule.

Rule 5.3. Find S triples of symmetric matrices Γ_i, Γ_{1i} and $\Gamma_{2i}(1 \le i \le S)$ with $\Gamma_i > 0$ such that for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d, i \in \mathbb{S}$

$$\begin{aligned} & 2(x-\bar{x})^{\mathrm{T}}\Gamma_i(f(x,y,i) - f(\bar{x},\bar{y},i)) \\ & + \mathrm{trace}\left((g(x,y,i) - g(\bar{x},\bar{y},i))^{\mathrm{T}}\Gamma_i(g(x,y,i) - g(\bar{x},\bar{y},i))\right) \\ \le & (x-\bar{x})^{\mathrm{T}}\Gamma_{1i}(x-\bar{x}) + (y-\bar{y})^{\mathrm{T}}\Gamma_{2i}(y-\bar{y}). \end{aligned} \tag{5.2}$$

Then for each $i \in \mathbb{S}$, design D_i such that there is a positive constant μ for the following two condition to hold

$$\tilde{\Gamma}_i := \Gamma_{1i} + D_i^{\mathrm{T}}\Gamma_i + \Gamma_i D_i + \sum_{j\in\mathbb{S}} q_{ij}\Gamma_j + \mu I_d < 0, \quad \forall i \in \mathbb{S}, \tag{5.3}$$

$$\varphi_0 := \frac{\mu}{\hat{\mu}_1} - \frac{\hat{\mu}_2}{\check{\mu}_1} > 0, \tag{5.4}$$

where $\check{\mu}_1 = \min_{i\in\mathbb{S}} \lambda_{\min}(\Gamma_i)$, $\hat{\mu}_1 = \max_{i\in\mathbb{S}} \lambda_{\max}(\Gamma_i)$, $\hat{\mu}_2 = \max_{i\in\mathbb{S}} \lambda_{\max}(\Gamma_{2i})$.

The reader might wonder if we can find Γ_i, Γ_{1i}, Γ_{2i} and D_i to let this rule be satisfied. The answer is positive under our Assumption 5.1. For condition (5.2), the simplest choice is to let $\Gamma_i = I_d, \Gamma_{1i} = (2L_1 + L_2 + 2L_3^2)I_d, \Gamma_{2i} = (L_2 + 2L_4^2)I_d$. Then we can choose

$$D_i = -2(L_1 + L_2 + L_3^2 + L_4^2)I_d, \quad \mu = L_1 + 2L_2 + L_3^2 + 3L_4^2, \quad \forall i \in \mathbb{S},$$

so that $\tilde{\Gamma}_i = -(L_1 + L_2 + L_3^2 + L_4^2)I_d < 0$ and $\varphi_0 = L_1 + L_2 + L_3^2 + L_4^2 > 0$. But it is wise to find alternative matrices in order to make use of the information of system coefficients. As for how to solve the linear matrix inequality (5.3), we refer the reader to [17] for more details.

Remark 5.4. It should also be highlighted the importance of Rule 5.3 on how to help the controlled SDDE (5.1) achieve the stability in distribution. Combining (5.2) with (5.3), we see that

$$\begin{aligned}
\bar{J}_1(x, \bar{x}, y, \bar{y}, i) :=& 2(x - \bar{x})^{\mathrm{T}}\Gamma_i(f(x, y, i) - f(\bar{x}, \bar{y}, i)) + (x - \bar{x})^{\mathrm{T}}(D_i^{\mathrm{T}}\Gamma_i + \Gamma_i D_i)(x - \bar{x}) \\
&+ \mathrm{trace}\big((g(x, y, i) - g(\bar{x}, \bar{y}, i))^{\mathrm{T}}\Gamma_i(g(x, y, i) - g(\bar{x}, \bar{y}, i))\big) \\
&+ \sum_{j\in\mathbb{S}} q_{ij}(x - \bar{x})^{\mathrm{T}}\Gamma_j(x - \bar{x}) \\
\le& -\mu|x - \bar{x}|^2 + \hat{\mu}_2|y - \bar{y}|^2
\end{aligned} \tag{5.5}$$

holds for any $(x, \bar{x}, y, \bar{y}, i) \in \mathbb{R}^{4d} \times \mathbb{S}$. If we take each Γ_i as I_d, then this becomes the attractivity Assumption 4.5. But where is the other key condition for the boundedness (e.g., Assumption 3.4)? Although Assumption 3.4 could not be deduce from Assumption 4.5 in general, this becomes possible under the linear growth condition. For any $(x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$, letting $\bar{x} = \bar{y} = 0$ in $\bar{J}_1(x, \bar{x}, y, \bar{y}, i)$ gives that

$$\begin{aligned}
J_1(x, y, i) :=& 2x^{\mathrm{T}}\Gamma_i f(x, y, i) + \mathrm{trace}\big(g^{\mathrm{T}}(x, y, i)\Gamma_i g(x, y, i)\big) + \sum_{j\in\mathbb{S}} q_{ij}x^{\mathrm{T}}\Gamma_j x \\
\le& -\mu|x|^2 + \hat{\mu}_2|y|^2 + 2|x^{\mathrm{T}}\Gamma_i f(0, 0, i)| + \big|\mathrm{trace}\big(g^{\mathrm{T}}(0, 0, i)\Gamma_i g(x, y, i) \\
&+ g^{\mathrm{T}}(x, y, i)\Gamma_i g(0, 0, i) - g^{\mathrm{T}}(0, 0, i)\Gamma_i g(0, 0, i)\big)\big| \\
\le& -\mu|x|^2 + \hat{\mu}_2|y|^2 + 2\|\Gamma_i\|K_3|x| + 2m\|\Gamma_i\|K_4(K_4 + L_3|x| + L_4|y|) \\
&+ m\|\Gamma_i\|K_4^2.
\end{aligned}$$

Choosing an appropriate $\epsilon > 0$, we could obtain that

$$J_1(x, y, i) \le C_0 - (\mu - \epsilon)|x|^2 + (\hat{\mu}_2 + \epsilon)|y|^2, \tag{5.6}$$

where C_0 is a positive constant defined by

$$\begin{aligned}
C_0 =& \sup_{i\in\mathbb{S}} \|\Gamma_i\|\Big(3mK_4^2 + \sup_{x\in\mathbb{R}^d}\big(-\epsilon|x|^2 + 2(mL_3K_4 + K_3)|x|\big) \\
&+ \sup_{y\in\mathbb{R}^d}\big(-\epsilon|y|^2 + 2mL_4K_4|y|\big)\Big).
\end{aligned}$$

As a result, we do not need to give additional rules to guarantee the boundedness.

Recalling the definition of $J_1(x, y, i)$ or $\bar{J}_1(x, \bar{x}, y, \bar{y}, i)$ above, we need only derive the stability in distribution of the controlled SDDE with feedback control $D_{r(t)}x(t)$

based on continuous-time observations. For our discrete-time feedback control, the differences $|x^{\xi,i_0}(t)-x^{\xi,i_0}(t-H(t))|$ and $|(x^{\xi,i_0}(t)-x^{\bar{\xi},\bar{i}_0}(t))-(x^{\xi,i_0}(t-H(t))-x^{\bar{\xi},\bar{i}_0}(t-H(t)))|$ should be limited. This could be achieved by the next rule on determining the observation duration, where we could see the importance of choosing matrices wisely in the rule above since they influence the value of τ.

Rule 5.5. Let $\varphi_1=\frac{D_\Gamma^2(4L_1+4L_2+4\hat{D}+2L_3^2+2L_4^2)}{\check{\mu}_1}$ with $D_\Gamma=\max_{i\in\mathbb{S}}(\|D_i^{\mathrm{T}}\Gamma_i+\Gamma_iD_i\|)$, $\hat{D}=\max_{i\in\mathbb{S}}\|D_i\|$. Then the observation duration τ should satisfy that

$$\tau<\tau^*:=\frac{\check{\mu}_1\varphi_0^2}{\varphi_1}. \tag{5.7}$$

Moreover, in order to form the time-homogeneous strong Markov process for the controlled SDDE (5.1), we will set τ as a divisor of h, that is, $\frac{h}{\tau}$ is an integer. In particular, if the delay $\delta(t)$ vanishes (namely, $h=0$), τ could be any positive integer satisfying (5.7).

In the remaining part of this section, let τ satisfy Rule 5.5. Since τ is a divisor of h, both $H(t)$ and $\delta(t)$ are periodic functions with period h. Then $\{x_{kh}^{\xi,i_0},r^{i_0}(kh)\}_{k\in\mathbb{N}_+}$ is also a time-homogeneous strong Markov process in $\mathbb{C}_h\times\mathbb{S}$. The techniques to prove the stability in distribution of the controlled SDDE (5.1) are similar to those in Section 4, where the key steps are to check $\sup_{0\le t<\infty}E|x^{\xi,i_0}(t)|^2<\infty$ and $\lim_{t\to\infty}E|x^{\xi,i_0}(t)-x^{\bar{\xi},i_0}(t)|^2\to 0$. They are stated in the following two lemmas respectively.

Lemma 5.6. *Let Assumption 5.1 and Rule 5.3 hold. Then the controlled SDDE (5.1) obeys that*

$$\sup_{0\le t<\infty}E|x^{\xi,i_0}(t)|^2<\infty,\quad \forall(\xi,i_0)\in\mathbb{C}_h\times\mathbb{S}. \tag{5.8}$$

Proof. Write $x(t)=x^{\xi,i_0}(t)$ and $r(t)=r^{i_0}(t)$ simply. Applying the generalized Itô formula to the function $\tilde{V}(x,i)=x^{\mathrm{T}}\Gamma_ix$ yields that

$$E\tilde{V}(x(t),r(t))=\tilde{V}(\xi(0),i_0)+\int_0^t E\tilde{L}\tilde{V}(x(s),x(s-\delta(s)),x(s-H(s)),r(s))\mathrm{d}s, \tag{5.9}$$

where $\tilde{L}\tilde{V}(x,y,z,i)=J_1(x,y,i)+J_2(x,z,i)$ with $J_2(x,z,i)=2x^{\mathrm{T}}\Gamma_iD_i(z-x)$. By virtue of Corollary 2.3, the remaining work is to estimate $E\tilde{L}\tilde{V}(x(t),x(t-\delta(t)),x(t-H(t)),r(t))$ in term of $E\tilde{V}(x(t),r(t))$ since the right-continuity is very clear.

Since $\frac{\varphi_1\tau}{\check{\mu}_1\varphi_0}<\varphi_0$ required in Rule 5.5, we can choose ϵ in (5.8) such that

$$\frac{\varphi_0}{2}-\frac{\varphi_1\tau}{2\check{\mu}_1\varphi_0}>\left(\frac{1}{\hat{\mu}_1}+\frac{1}{\check{\mu}_1}+\frac{3D_\Gamma^2\tau}{2\check{\mu}_1^2\varphi_0}\right)\epsilon. \tag{5.10}$$

Then the estimate of EJ_1 is a direct result from (5.6)

$$\begin{aligned}&EJ_1(x(t),x(t-\delta(t)),r(t))\\ \le&C_0-\frac{\mu-\epsilon}{\hat{\mu}_1}E\tilde{V}(x(t),r(t))+\frac{\hat{\mu}_2+\epsilon}{\check{\mu}_1}E\tilde{V}(x(t-\delta(t)),r(t-\delta(t))).\end{aligned} \tag{5.11}$$

As for EJ_2, we see that for any $(x,z,i)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}$

$$J_2(x,z,i)=x^{\mathrm{T}}(D_i^{\mathrm{T}}\Gamma_i+\Gamma_iD_i)(z-x)\le\frac{\check{\mu}_1\varphi_0}{2}|x|^2+\frac{D_\Gamma^2}{2\check{\mu}_1\varphi_0}|x-z|^2,$$

which implies that

$$EJ_2(x(t),x(t-H(t)),r(t)) \le \frac{\varphi_0}{2}E\tilde{V}(x(t),r(t))+\frac{D_\Gamma^2}{2\check{\mu}_1\varphi_0}E|x(t)-x(t-H(t))|^2. \tag{5.12}$$

For any $t\in\mathbb{R}_+$, there is an integer n such that $n\tau\le t<(n+1)\tau$. Thus $x(t)-x(t-H(t))=x(t)-x(n\tau)$ is in fact, an Itô process on $[n\tau,(n+1)\tau]$, and also for any $s\in[n\tau,t]$, $x(s-H(s))=x(n\tau)$. Applying the Itô formula to $|x|^2$ and using the linear growth conditions on f and g, we have

$$\begin{aligned}
&E|x(t)-x(n\tau)|^2\\
=&\int_{n\tau}^t \Big(2(x(s)-x(n\tau))^{\mathrm{T}}(f(x(s),x(s-\delta(s)),r(s))+D_{r(s)}x(n\tau)\Big)\\
&+|g(x(s),x(s-\delta(s)),r(s))|^2\Big)\mathrm{d}s\\
\le& C_1\tau+E\int_{t-\tau}^t\Big((3L_1+L_2+2L_3^2+\hat{D}+\epsilon)|x(s)|^2+(L_1+L_2+3\hat{D}+\epsilon)|x(n\tau)|^2\\
&+(2L_2+2L_4^2+\epsilon)|x(s-\delta(s))|^2\Big)\mathrm{d}s\\
\le& C_1\tau+\frac{4L_1+4L_2+4\hat{D}+2L_3^2+2L_4^2+3\epsilon}{\check{\mu}_1}\tau\sup_{-h-\tau\le\theta\le0}E\tilde{V}(x(t+\theta),r(t+\theta)),
\end{aligned}$$

where

$$\begin{aligned}
C_1=&K_4^2+\sup_{x\in\mathbb{R}^d}\big((2K_3+2K_4L_3)|x|-\epsilon|x|^2\big)+\sup_{x\in\mathbb{R}^d}\big(2K_3|x|-\epsilon|x|^2\big)\\
&+\sup_{x\in\mathbb{R}^d}\big(2K_4L_4|x|-\epsilon|x|^2\big).
\end{aligned}$$

Substituting the estimation of $E|x(t)-x(n\tau)|^2$ into (5.12) gives that

$$\begin{aligned}
EJ_2(x(t),x(t-H(t)),r(t))\le&\frac{D_\Gamma^2C_1\tau}{2\check{\mu}_1\varphi_0}+\frac{\varphi_0}{2}E\tilde{V}(x(t),r(t))+\left(\frac{\varphi_1\tau}{2\check{\mu}_1\varphi_0}+\frac{3D_\Gamma^2\tau}{2\check{\mu}_1^2\varphi_0}\epsilon\right)\\
&\times\sup_{-h-\tau\le\theta\le0}E\tilde{V}(x(t+\theta),r(t+\theta)).
\end{aligned}\tag{5.13}$$

Finally, putting (5.11) and (5.13) into $E\tilde{L}\tilde{V}$, we obtain that

$$\begin{aligned}
&E\tilde{L}\tilde{V}(x(t),x(t-\delta(t)),x(t-H(t)),r(t))\\
\le& C_0+\frac{D_\Gamma^2C_1\tau}{2\check{\mu}_1\varphi_0}-\left(\frac{\mu-\epsilon}{\hat{\mu}_1}-\frac{\varphi_0}{2}\right)E\tilde{V}(x(t),r(t))\\
&+\left(\frac{\hat{\mu}_2+\epsilon}{\check{\mu}_1}+\frac{\varphi_1\tau}{2\check{\mu}_1\varphi_0}+\frac{3D_\Gamma^2\tau}{2\check{\mu}_1^2\varphi_0}\epsilon\right)\sup_{-h-\tau\le\theta\le0}E\tilde{V}(x(t+\theta),r(t+\theta)).
\end{aligned}\tag{5.14}$$

From the definition of ϵ in (5.10), it is easy to see that

$$\begin{aligned}
&\left(\frac{\mu-\epsilon}{\hat{\mu}_1}-\frac{\varphi_0}{2}\right)-\left(\frac{\hat{\mu}_2+\epsilon}{\check{\mu}_1}+\frac{\varphi_1\tau}{2\check{\mu}_1\varphi_0}+\frac{3D_\Gamma^2\tau}{2\check{\mu}_1^2\varphi_0}\epsilon\right)\\
=&\frac{\varphi_0}{2}-\frac{\varphi_1\tau}{2\check{\mu}_1\varphi_0}-\left(\frac{1}{\hat{\mu}_1}+\frac{1}{\check{\mu}_1}+\frac{3D_\Gamma^2\tau}{2\check{\mu}_1^2\varphi_0}\right)\epsilon>0.
\end{aligned}$$

The proof is completed. ∎

Next, we show the attractivity of solutions under the same settings.

LEMMA 5.7. *Let all the conditions in Lemma 5.6 hold. Then for any positive number R,*

$$\lim_{t\to\infty} E|x^{\xi,i_0}(t) - x^{\bar{\xi},i_0}(t)|^2 \to 0, \quad \textit{uniformly in} \;\; \xi, \bar{\xi} \in \mathbb{B}_R. \tag{5.15}$$

Proof. The proof process is quite similar to Lemma 4.6, so we only highlight the difference. The key step is to obtain

$$\lim_{t\to\infty} E|x(t) - \bar{x}(t)|^2 \to 0, \quad \forall \xi, \bar{\xi} \in \mathbb{B}_R.$$

The notations $x(t)$ and $\bar{x}(t)$ keep the same as in Lemma 4.6. To show this claim, we need to focus on the following equation

$$\begin{aligned}
\mathrm{d}(x(t) - \bar{x}(t)) = & \left(f(x(t), x(t-\delta(t)), r(t)) - f(\bar{x}(t), \bar{x}(t-\delta(t)), r(t))\right)\mathrm{d}t \\
& + D_{r(t)}(x(t-H(t)) - \bar{x}(t-H(t)))\mathrm{d}t \\
& + \left(g(x(t), x(t-\delta(t)), r(t)) - g(\bar{x}(t), \bar{x}(t-\delta(t)), r(t))\right)\mathrm{d}W(t).
\end{aligned}$$

Just as we do in Lemma 5.6, after applying the generalized Itô formula to $\tilde{U}(x,i) = x^{\mathrm{T}}\Gamma_i x$, the only work is to estimate $\tilde{L}\tilde{U} : \mathbb{R}^{6d} \times \mathbb{S} \to \mathbb{R}$ given by

$$\begin{aligned}
& \tilde{L}\tilde{U}(x, \bar{x}, y, \bar{y}, z, \bar{z}, i) \\
= & 2(x-\bar{x})^{\mathrm{T}}\Gamma_i(f(x,y,i) - f(\bar{x},\bar{y},i)) + (x-\bar{x})^{\mathrm{T}}(D_i^{\mathrm{T}}\Gamma_i + \Gamma_i D_i)(z - \bar{z}) \\
& + \operatorname{trace}\left((g(x,y,i) - g(\bar{x},\bar{y},i))^{\mathrm{T}}\Gamma_i(g(x,y,i) - g(\bar{x},\bar{y},i))\right) \\
& + \sum_{j\in\mathbb{S}} q_{ij}(x-\bar{x})^{\mathrm{T}}\Gamma_j(x-\bar{x}).
\end{aligned}$$

From (5.5), compute

$$\begin{aligned}
& E\tilde{L}\tilde{U}(x(t), \bar{x}(t), x(t-\delta(t)), \bar{x}(t-\delta(t)), x(t-H(t)), \bar{x}(t-H(t)), r(t)) \\
\leq & -\left(\frac{\mu}{\hat{\mu}_1} - \frac{\varphi_0}{2}\right) E\tilde{U}(x(t) - \bar{x}(t), r(t)) + \frac{\hat{\mu}_2}{\check{\mu}_1} E\tilde{U}(x(t-\delta(t)) - \bar{x}(t-\delta(t)), r(t-\delta(t))) \\
& + \frac{D_\Gamma^2}{2\check{\mu}_1\varphi_0} E|(x(t-H(t)) - \bar{x}(t-H(t))) - (x(t) - \bar{x}(t))|^2.
\end{aligned}$$

Noting the constant property of $t - H(t)$, we have $n\tau \leq t - H(t) < (n+1)\tau$ and then

$$\begin{aligned}
& E|(x(t) - \bar{x}(t)) - (x(n\tau) - \bar{x}(n\tau))|^2 \\
= & \int_{n\tau}^{t} \Bigg(2\Big((x(s) - \bar{x}(s)) - (x(n\tau) - \bar{x}(n\tau))\Big)^{\mathrm{T}} \\
& \times \Big(f(x(s), x(s-\delta(s)), r(s)) - f(\bar{x}(s), \bar{x}(s-\delta(s)), r(s)) + D_{r(s)}(x(n\tau) - \bar{x}(n\tau))\Big) \\
& + 2|g(x(s), x(s-\delta(s)), r(s)) - g(\bar{x}(s), \bar{x}(s-\delta(s)), r(s))|^2 \Bigg)\mathrm{d}s \\
\leq & E\int_{t-\tau}^{t} \Big((3L_1 + L_2 + 2L_3^2 + \hat{D})|x(s) - \bar{x}(s)|^2 + (L_1 + L_2 + 3\hat{D})|x(n\tau) - \bar{x}(n\tau)|^2 \\
& + (2L_2 + 2L_4^2)|x(s-\delta(s)) - \bar{x}(s-\delta(s))|^2 \Big)\mathrm{d}s
\end{aligned}$$

$$\leq \frac{4L_1 + 4L_2 + 4\hat{D} + 2L_3^2 + 2L_4^2}{\check{\mu}_1}\tau \sup_{-h-\tau\leq\theta\leq 0} \Phi(t+\theta).$$

Substituting this into $E\tilde{L}\tilde{U}$ gives that

$$\begin{aligned} &E\tilde{L}\tilde{U}(x(t),\bar{x}(t),x(t-\delta(t)),\bar{x}(t-\delta(t)),x(t-H(t)),\bar{x}(t-H(t)),r(t)) \\ &\leq -\left(\frac{\mu}{\hat{\mu}_1} - \frac{\varphi_0}{2}\right) E\tilde{U}(x(t)-\bar{x}(t),r(t)) \\ &\quad + \left(\frac{\hat{\mu}_2}{\check{\mu}_1} + \frac{\varphi_1\tau}{2\check{\mu}_1\varphi_0}\right) \sup_{-h-\tau\leq\theta\leq 0} E\tilde{U}(x(t+\theta)-\bar{x}(t+\theta),r(t+\theta)). \end{aligned} \tag{5.16}$$

From Rule 5.5, we could see that $\left(\frac{\mu}{\hat{\mu}_1} - \frac{\varphi_0}{2}\right) - \left(\frac{\hat{\mu}_2}{\check{\mu}_1} + \frac{\varphi_1\tau}{2\check{\mu}_1\varphi_0}\right) > 0$. Following the procedure in Lemma 4.6 could imply the required (5.15). □

Using the same method of proving Lemmas 4.4, 4.7, we could see from Lemma 5.6 that property (P1) holds, and property (P2) from Lemma 5.7. The following conclusion could hence be drawn easily.

THEOREM 5.8. *Let all the conditions in Lemma 5.6 hold. Then the controlled SDDE (5.1) is stable in distribution. In other words, there is a probability measure π in $\mathbb{P}(\mathbb{C}_h)$ such that $\lim_{k\to\infty} d_{\mathbb{C}}(\mathcal{P}(x_{kh}^{\xi,i_0}),\pi) = 0$. Further, for any $h^0 \in [0,h)$, there is a unique $\pi_{h^0} \in \mathbb{P}(\mathbb{R}^d)$ such that $\lim_{k\to\infty} d_{\mathbb{R}}(\hat{\mathcal{P}}(x^{\xi,i_0}(h^0+kh)),\pi_{h^0}) = 0$.*

6. Illustrative examples.

6.1. Example 1. FitzHugh-Nagumo (FHN) system is a well-known nonlinear model of a spiking neuron, designed to capture the essential dynamics of action potential generation in excitable cells and cardiac cells [7]. Consider an FHN system with Markovian switching by

$$\begin{cases} \dot{x}_1(t) = x_1(t) - \frac{x_1^3(t)}{3} - x_2(t) + I_{ext}, \\ \dot{x}_2(t) = \hat{\epsilon}_{r(t)}(x_1(t) + \nu_{1,r(t)} - \nu_{2,r(t)}x_2(t)), \end{cases} \tag{6.1}$$

where $x_1(t)$ and $x_2(t)$ represent the membrane potential and recovery variable, respectively, $\hat{\epsilon}_i$ controls the speed of recovery, $\nu_{1,i}$, $\nu_{2,i}$ are constants of recovery process' dynamics, I_{ext} is the external stimulation current like signal input. In a complex environment, the input is not only made based on current-time states, but sometimes depends on sampling states and noises, such as the following form

$$-\Big(R_{1,r(t)} + \sigma_{1,r(t)}\dot{W}_1(t)\Big)x_1(t) - \Big(R_{2,r(t)} + \sigma_{2,r(t)}\dot{W}_2(t)\Big)x_1([t/h]h) + \sigma_{3,r(t)}\dot{W}_3(t).$$

For simplicity, let $r(t)$ be a Markov chain taking values in $\mathbb{S} = \{1,2\}$ with generator $Q = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$, $W_1(t), W_2(t), W_3(t)$ be scalar Brownian motions independent of each other. The parameters are given as $h = 0.02$, $R_{1,1} = 5$, $R_{1,2} = 3$, $R_{2,1} = 0.5$, $R_{2,2} = 0.1$, $\sigma_{1,1} = 0.2$, $\sigma_{1,2} = 0.1$, $\sigma_{2,1} = 0.4$, $\sigma_{2,2} = 0.2$, $\sigma_{3,1} = 1$, $\sigma_{3,2} = 0.5$, $\hat{\epsilon}_1 = \hat{\epsilon}_2 = 0.1$, $\nu_{1,1} = 1$, $\nu_{1,2} = 0.3$, $\nu_{2,1} = 15$, $\nu_{2,2} = 10$.

Letting $x(t) = (x_1(t),x_2(t))^{\mathrm{T}}, W(t) = (W_1(t),W_2(t),W_3(t))^{\mathrm{T}}$, we can rewrite FHN system (6.1) as a general form of hybrid SDDE

$$\mathrm{d}x(t) = F(x(t),x([t/h]h),r(t))\mathrm{d}t + G(x(t),x([t/h]h),r(t))\mathrm{d}W(t), \tag{6.2}$$

where

$$F(x,y,i)=\begin{pmatrix}-(R_{1,i}-1)x_1-x_2-R_{2,i}y_1-\frac{x_1^3}{3}\\ \hat{\epsilon}_i\nu_{1,i}+\hat{\epsilon}_i x_1-\hat{\epsilon}_i\nu_{2,i}x_2\end{pmatrix},$$

$$G(x,y,i)=\begin{pmatrix}-\sigma_{1,i}x_1 & -\sigma_{2,i}y_1 & \sigma_{3,i}\\ 0 & 0 & 0\end{pmatrix}.$$

It is easy to calculate for every $x,y\in\mathbb{R}^2, i\in\mathbb{S}$,

$$|F(x,y,i)|^2\le 3\hat{\epsilon}_i^2\nu_{1,i}^2+(3(R_{1,i}-1)^2+3\hat{\epsilon}_i^2(1+\nu_{2,i}^2))|x|^2+3R_{2,i}^2|y|^2+\frac{1}{2}|x|^6,$$

$|G(x,y,i)|^2=3\sigma_{1,i}^2|x|^2+3\sigma_{2,i}^2|y|^2+3\sigma_{3,i}^2$ and

$$\begin{aligned}x^{\mathrm{T}}F(x,y,i)\le&-\left(\left(R_{1,i}-1-\frac{R_{2,i}}{2}-\frac{\hat{\epsilon}_i+1}{2}\right)\wedge\left(0.99\hat{\epsilon}_i\nu_{2,i}-\frac{\hat{\epsilon}_i+1}{2}\right)\right)|x|^2\\&+\frac{R_{2,i}}{2}|y|^2+C_F,\end{aligned}$$

with $C_F=\sup_{a\in\mathbb{R}}(\hat{\epsilon}_i\nu_{1,i}a-0.01\hat{\epsilon}_i\nu_{2,i}a^2)$. For $x,\bar{x},y,\bar{y}\in\mathbb{R}^2$

$$\begin{aligned}&(x-\bar{x})^{\mathrm{T}}(F(x,y,i)-F(\bar{x},\bar{y},i))\\ \le&-\left(\left(R_{1,i}-1-\frac{R_{2,i}}{2}-\frac{\hat{\epsilon}_i+1}{2}\right)\wedge\left(\hat{\epsilon}_i\nu_{2,i}-\frac{\hat{\epsilon}_i+1}{2}\right)\right)|x-\bar{x}|^2+\frac{R_{2,i}}{2}|y-\bar{y}|^2,\end{aligned}$$

and $|G(x,y,i)-G(\bar{x},\bar{y},i)|^2=2\sigma_{1,i}^2|x-\bar{x}|^2+2\sigma_{2,i}^2|y-\bar{y}|^2$. Therefore, Assumption 3.1 holds with $K_0=1.7321$, $K_1=8.5282$, $K_2=0.6928$, $p=3$, Assumption 3.2 holds with $\alpha=4.8333$. Assumption 3.4 is satisfied with

$$\alpha_{1,1}=4.8333,\ \alpha_{1,2}=2.25,\ \alpha_{2,1}=-0.875,\ \alpha_{2,2}=-0.425,\ \alpha_{3,1}=0.49,\ \alpha_{3,2}=0.11,$$

$A=\begin{pmatrix}3.75 & -2\\ -1 & 1.85\end{pmatrix}$, $(\eta_1,\eta_2)=(0.7797,0.962)$ so that $\frac{\check{\eta}}{\hat{\eta}}-2\max_{i\in\mathbb{S}}(\eta_i\alpha_{3i})=0.0464>0$. Assumption 4.5 is satisfied with

$$\beta_{2,1}=-0.89,\quad \beta_{2,2}=-0.435,\quad \beta_{3,1}=0.49,\quad \beta_{3,2}=0.11,$$

$B=\begin{pmatrix}3.78 & -2\\ -1 & 1.87\end{pmatrix}$, $(\gamma_1,\gamma_2)=(0.7635,0.9431)$ so that $\frac{\check{\gamma}}{\hat{\gamma}}-2\max_{i\in\mathbb{S}}(\gamma_i\beta_{3i})=0.0614>0$. By Theorem 4.8 and Corollary 4.9, FHN (6.2) is stable in distribution and $\{x(h^0+kh)\}_{k\in\mathbb{N}_+}$ also converges to a stationary distribution for any $h^0\in[0,h)$.

For simulation, we first use the Euler-Marayama method with step size $\Delta=0.001$ to obtain 5000 sample paths. Then choose $h^0=0$ and do energy-test (see [29]) between $\mathbb{R}^2$-valued samples $x(kh)$ and $x((k+1)h)$ for $k=0,1,\cdots$. The result is shown in Fig.1 The upper is the E-statistics (energy distances), the values of which are close to zero means two samples are almost from the same distributions. This shows that the probability distribution of $x(kh)$ will remain unchangeable as k advances. The lower is the p values of each test, which are larger than 0.6 for large time, additionally indicating that we cannot reject that distributions at quite distant time points follow the same distribution. In other words, $\{x(kh)\}_{k\in\mathbb{N}_+}$ will converge to a stationary distribution.

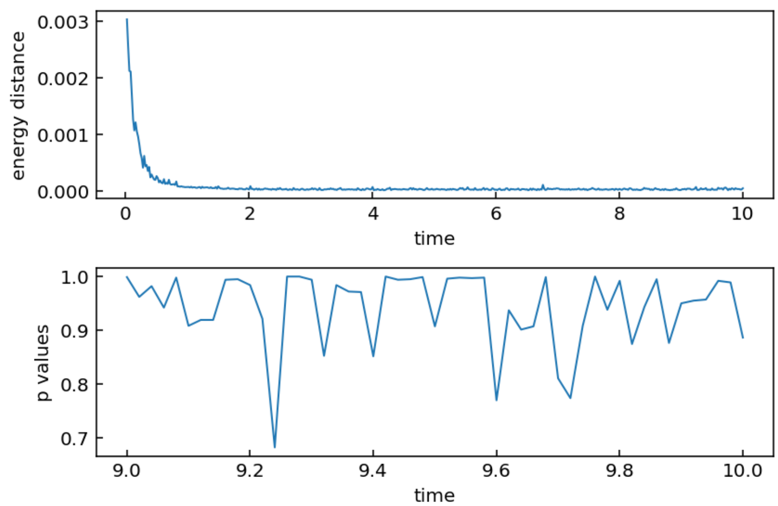

FIG. 1. *Energy distances and p values over time of FHN(6.2).*

6.2. Example 2. Next we use this example to show that our developed theory could improve the estimation of upper bound of τ. Consider the same example studied in [17], namely, the 2-dimensional hybrid SDE

$$\mathrm{d}x(t) = \big(b_{1,r(t)} + \mathcal{B}_{1,r(t)}x(t)\big)\,\mathrm{d}t + \big(b_{2,r(t)} + \mathcal{B}_{2,r(t)}x(t)\big)\,\mathrm{d}W(t). \tag{6.3}$$

This is a special case that $\delta(t) \equiv 0$. Here $W(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain independent of $W(t)$ with state space $\mathbb{S} = \{1, 2\}$ and generator $Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$. The system parameters are given by $b_{1,1} = (10, 5)^{\mathrm{T}}$, $b_{1,2} = (15, 10)^{\mathrm{T}}$, $b_{2,1} = (0.5, 0.3)^{\mathrm{T}}$, $b_{2,2} = (0.4, 0.6)^{\mathrm{T}}$,

$$\mathcal{B}_{1,1} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{B}_{2,1} = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}, \quad \mathcal{B}_{3,1} = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & -0.1 \end{pmatrix}, \quad \mathcal{B}_{4,1} = \begin{pmatrix} -0.1 & -0.1 \\ -0.1 & 0.1 \end{pmatrix}.$$

From the computer simulations in [17], hybrid SDE (6.3) is not stable in distribution. We aim to find a linear control based on discrete-time observations to make the controlled SDE

$$\mathrm{d}x(t) = \big(b_{1,r(t)} + \mathcal{B}_{1,r(t)}x(t) + D_{r(t)}x([t/\tau]\tau)\big)\,\mathrm{d}t + \big(b_{2,r(t)} + \mathcal{B}_{2,r(t)}x(t)\big)\,\mathrm{d}W(t), \tag{6.4}$$

stable in distribution. We still use the same control function in [17], that is

$$D_1 = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}. \tag{6.5}$$

It is easy to verify that Assumption 5.1 holds with $L_1 = 3.1623$, $L_2 = 0$, $L_3 = 0.2$, $L_4 = 0$, $K_3 = 18.0278$, $K_4 = 0.7211$. To verify Rule 5.3, we could take Γ_1, Γ_2 as 2×2 identity matrix, $\Gamma_{21} = \Gamma_{22} = 0$, and $\Gamma_{1i} = \Gamma_i \mathcal{B}_{1,i} + \mathcal{B}_{1,i}^{\mathrm{T}} + 2\mathcal{B}_{2,i}^{\mathrm{T}} \Gamma_i \mathcal{B}_{2,i}$, that is

$$\Gamma_{11} = \begin{pmatrix} 4.04 & 0 \\ 0 & -1.96 \end{pmatrix}, \quad \Gamma_{12} = \begin{pmatrix} -1.96 & 0 \\ 0 & 2.04 \end{pmatrix}.$$

We choose $\mu = 1.95$ and D_i in (6.5) to see that $\tilde{\Gamma}_1 = \tilde{\Gamma}_2 = \begin{pmatrix} -0.01 & 0 \\ 0 & -0.01 \end{pmatrix}$ and $\varphi_0 = 1.95$. By Rule 5.5, compute $\tau < 0.1538$, which is better than the result in [17], where τ should be less than 0.01746. Since $h = 0$ in this special case, we could then choose $\tau = 0.1$ for simulation. The result of energy test is given in Fig.2 (where the step size is 0.01, and we generate 5000 sample paths and with time point difference being 0.1). It demonstrates the convergence of the distributions of sequence $\{x(k\tau)\}_{k\in\mathbb{N}_+}$ from the controlled SDE (6.4).

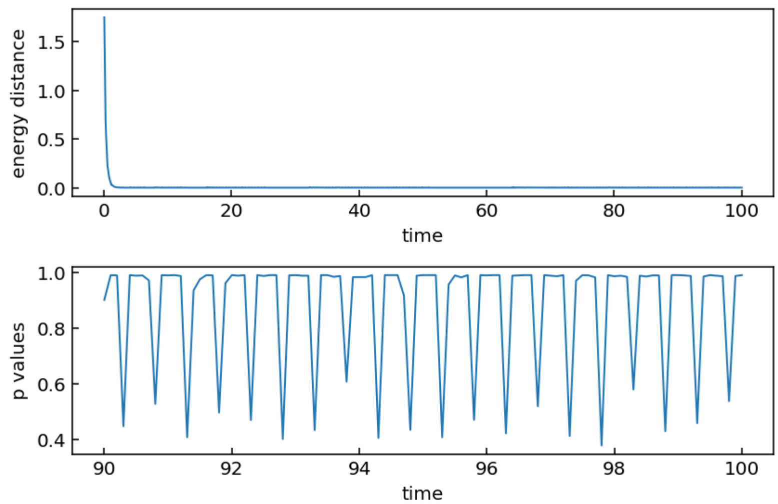

FIG. 2. *Energy distances and p values over time of the controlled SDE (6.4).*

7. Conclusion. In this paper, we successfully used the generalized Halanay inequality described by right-continuous functions to develop moment boundedness, moment exponential stability of equilibrium points, and asymptotic stability in distribution for hybrid SDDEs with very general time-variable delays. Stabilization by discrete-time feedback control was also studied if a given hybrid SDDE was not stable in distribution. The upper bound of observation duration could be obtained by a linear equation, and its value could be improved compared with the existing results. The application to FitzHugh-Nagumo system showed the practical feasibility of our theory.

In the future, we will consider a more general type of the Halanay inequality, whose right-hand side is governed by nonlinear form of the objective function. As a result, Assumption 3.4 could be relaxed to the generalized Khasminskii-type condition for hybrid SDDEs. Besides, more techniques should be developed to the estimation of $|(x^{\xi,i_0}(t) - x^{\bar{\xi},\bar{i}_0}(t)) - (x^{\xi,i_0}(t-H(t)) - x^{\bar{\xi},\bar{i}_0}(t-H(t)))|$, and so the global Lipschitz condition can be relaxed from the study of discrete-state-feedback stabilization in distribution.

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