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Analyzing the asymptotic behavior and ergodicity of a stochastic boycotting commercial product model.

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Abstract

In this work, we apply stochasticity to the PBC model, which characterizes citizens' product boycotting behavior, where individuals are potential consumers, boycotters who refrain from buying it for a variety of reasons, and current consumers. The standard technique in stochastic is used in stochasticity in population modeling. We study the existence and uniqueness of the nonnegative solution. Moreover, we investigate the asymptotic behavior of this solution, and we conclude that the boycott will succeed and prevail by the ergodic property's solution and its stationary distribution. Lastly, some Matlab numerical simulations are run to confirm the theoretical analysis.

Keywords: Asymptotic behavior, Ergodic stationary distribution, Stochastic Lyapunov functions, Boycott-free equilibrium, Itô's formula.

1 Introduction

A boycott usually consists of a promise not to purchase anything from a firm because of a particular ethical concern.
The term "boycott" has been in use since the 1880s. Charles Boycott, a retired British army captain and estate manager, first used it when he attempted to evict farmers who couldn't pay their rent. His employees shunned him in protest. The

term "boycott" was used to describe the practice of avoiding goods or services from businesses that promote specific viewpoints or act in ways that contradict them.
Boycotts are used in labor movements to express disapproval of a variety of issues, including union-busting, poor pay, hazardous working conditions, excessive hours, and discriminatory policies. A boycott is frequently employed in conjunction with other strategies or as a last resort in union conflicts after all other strategies have failed. To different degrees of success, unions use the purchasing power of consumers to exert pressure on businesses through decreased sales and negative publicity. Just as social media and word-of-mouth are important in spreading boycott messages, so is ephemera. Among the useful resources that "go where organizers cannot always go" are buttons, stickers, flyers, posters, t-shirts, hats, and bumper stickers. (2019, Brantley) [9].
Successful boycotts in the past have targeted firms like Amazon for their egregious tax evasion, Barclays Bank for their interests in fossil fuels, and L'Oreal cosmetics for their policy against animal testing, among many others.
The boycott of South Africa in opposition to apartheid was arguably the most well-known. As a method of protest against racism, violence against Black people, and racial segregation imposed by white colonial officials, South African exiles and their allies called for a boycott of South African products in 1959. Initially limited to fruit and vegetables, the boycott later spread to chains like Marks & Spencer and Next, which caused several companies to cut South African goods out of their stock. Throughout the following 35 years, the boycott was crucial to the anti-apartheid campaign. Due to pressure from international leaders and decades of grassroots organizing, apartheid ended in 1994 [8].

Though there are still not many available mathematical studies and research on this topic [10–15], this and related topics have been the focus of several research initiatives and studies in the social, economic, and political sciences [16–22].
This work is regarded as an expansion of these earlier studies exactly on our work [10] which is interested in understanding the behavior of boycotting a product and suggested optimal strategies for controlling this behavior and stopping its spread, protecting product marketing and promoting customer reuse.
We presented a mathematical model that characterizes how citizens behave when it comes to consuming a particular product [10]. People are classified into three main groups in this model: those who are potentially consumers of the product, those who boycott it and try to persuade others to follow suit, and the group of consumers who are already using it, and consider the following PBC model :

$$
\begin{cases}
\dfrac{dP(t)}{dt} = A - k\dfrac{P(t)B(t)}{N} - (\mu+\lambda)P(t), \\[2ex]
\dfrac{dB(t)}{dt} = k\dfrac{P(t)B(t)}{N} - (\delta+\mu)B(t), \\[2ex]
\dfrac{dC(t)}{dt} = \lambda P(t) + \delta B(t) - \mu C(t).
\end{cases}
\tag{1}
$$

The following describes the parameters, which are all positive constants. N indicates the total population, which is divided into three compartments; the number of people who can access, buy, and consume the product is represented by potential consumers (P), which is growing at a rate A. It decreases at rate λ when potential consumers become actual consumers (C). Potential consumers can also become product boycotters (B) at rate k through meaningful interactions with current boycotters; at rate δ, those boycotters change their opinions about the product and become new actual consumers. The three compartments (P), (B), and (C) decrease at the rate μ of natural death [10].

Our analysis is predicated on a widely recognized approach to address the phenomenon of boycotting a product. The technique is often applied in models of infections diseases. It looks for a threshold value, known the basic reproduction number, that establishes if the equilibrium is stable. In system (1) the threshold is $R_0 = \dfrac{\mu k}{(\lambda+\mu)(\delta+\mu)}$. There is always a boycott-free equilibrium $E_0 = \left(\dfrac{A}{\lambda+\mu}, 0, \dfrac{\lambda A}{\mu(\lambda+\mu)}\right)$. Because the boycott-free equilibrium E_0 is globally asymptotically stable when $R_0 \le 1$, it becomes unstable when $R_0 > 1$, giving rise to boycotted equilibrium $E^* = (P^*, B^*, C^*) = \left(\dfrac{A}{(\lambda+\mu)R_0}, \dfrac{A(\lambda+\mu)(R_0-1)}{\mu k}, \dfrac{\lambda A(\delta+\mu)^2 + \delta A(\lambda+\mu)(\delta+\mu)(R_0-1)}{\mu^2(\delta+\mu)k}\right)$ which is globally asymptotically stable [10].

Since unpredictability and stochasticity are common in real life, there are actually substantial advantages to using stochastic models. On base of system (1), model (2) adds a random perturbation by switching the parameter λ with $\lambda \to \lambda + \sigma \dot{W}_i$, $i = 1, 2, 3$, where $\dot{W}_i$ is the white noise, specifically, $W_i(t)$ are independent standard Brownian motions and $\sigma_i^2 > 0$ indicates the white noise's intensity. The model takes the following form:

$$
\begin{cases}
dP(t) = \left[A - k\dfrac{P(t)B(t)}{N} - (\mu+\lambda)P(t)\right] dt + \sigma_1 P(t) dW_1(t), \\[2ex]
dB(t) = \left[k\dfrac{P(t)B(t)}{N} - (\delta+\mu)B(t)\right] dt + \sigma_2 B(t) dW_2(t), \\[2ex]
dC(t) = [\lambda P(t) + \delta B(t) - \mu C(t)]\, dt + \sigma_3 C(t) dW_3(t).
\end{cases}
\tag{2}
$$

For the rest of the article we consider that N is a constant $N = \frac{A}{\mu}$, taken from the interval $\Omega = \{(P(t), B(t), C(t)) \in \mathbb{R}^3_+, N(t) = P(t) + B(t) + C(t) \leq \frac{A}{\mu}\}$ in our basic article [10], so the system (2) will be (3).

$$\begin{cases} dP(t) = \left[A - \frac{k\mu}{A}P(t)B(t) - (\mu+\lambda)P(t)\right]dt + \sigma_1 P(t)dW_1(t), \\ dB(t) = \left[\frac{k\mu}{A}P(t)B(t) - (\delta+\mu)B(t)\right]dt + \sigma_2 B(t)dW_2(t), \\ dC(t) = \left[\lambda P(t) + \delta B(t) - \mu C(t)\right]dt + \sigma_3 C(t)dW_3(t). \end{cases} \tag{3}$$

Our approach is frequently used in this study to incorporate stochastic perturbation in stochastic modeling, as in [23–25]. it is organized as follows. In section 2, the global existence and positivity of the solution to system (3) are investigated. In section 3, we discuss asymptotic behavior of the boycott-free equilibrium E_0, and ergodicity of system (3) in section 4. Subsequently, numerical simulations are shown in section 5, the paper comes to its end in section 6.

2 Existence and Uniqueness of the Nonnegative Solution

2.1 Preliminary

Given a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$, let $\{\mathcal{F}\}_{t\geq 0}$ be a filtration that satisfies the usual conditions (i.e., it is right continuous and rising, whereas $\mathcal{F}_0$ contains all P-null sets). Additionally, let $x(t) = (P(t), B(t), C(t))^T$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n, x_i > 0$ for all $1 \leq i \leq n\}$.
The auxiliary assertions listed below were first presented in [7]. Consider the stochastic differential equation in d dimensions.

$$dx(t) = f(x(t), t)dt + g(x(t), t)dW(t), t \geq t_0. \tag{4}$$

The family of all nonnegative functions $\mathcal{V}(x, t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ that are continuously twice differentiable in `x` and once in `t` is denoted as $C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$. The formula defines the differential operator $\mathcal{L}$ of Equation (4) [7].

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x, t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(x, t)g(x, t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_i}.$$

When $\mathcal{L}$ impacts on a function $\mathcal{V} \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$, then

$$\mathcal{L}\mathcal{V}(x, t) = \mathcal{V}_t(x, t) + \mathcal{V}_x(x, t)f(x, t) + \frac{1}{2}trace[g^T(x, t)\mathcal{V}_{xx}(x, t)g(x, t)],$$

where $\mathcal{V}_t = \dfrac{\partial \mathcal{V}}{\partial t}$, $\mathcal{V}_x = \left(\dfrac{\partial \mathcal{V}}{\partial x_1}, \ldots, \dfrac{\partial \mathcal{V}}{\partial x_d}\right)$ and $\mathcal{V}_{xx} = \left(\dfrac{\partial^2}{\partial x_i \partial x_i}\right)_{d\times d}$.

Using Itô's formula,

$$d\mathcal{V}(x(t),t) = \mathcal{L}\mathcal{V}(x(t),t)dt + \mathcal{V}_x(x(t),t)g(x(t),t)dW(t).$$

2.2 Existence of unique positive solution

It makes sense to take into account the solution of the stochastic model (3) in the set $\Omega = \left\{(P(t),B(t),C(t)) \in \mathbb{R}^3_+, P(t)+B(t)+C(t) \leq \dfrac{A}{\mu}\right\}$ [10] since $P(t)$, $B(t)$ and $C(t)$ represent the number of potential customers, boycotters, and actual consumers. Therefore, we must demonstrate that the stochastic model (3) has a unique global solution and that the solution will always stay inside Ω whenever it begins there in order for it to make sense. After that, we determine the following theorem.

Theorem 1. *There is a unique solution $(P_0, B_0, C_0) \in \Omega$ of system (3) on $t \geq 0$ with probability 1, which is $\mathbb{P}\{P(t), B(t), C(t) \in \Omega, \forall t \geq 0\} = 1$, for any initial value $(P_0, B_0, C_0) \in \Omega$.*

Proof. For any initial value $(P_0, B_0, C_0) \in \Omega$, the coefficients of the equation are locally Lipschitz continuous, hence there is unique local solution $(P(t), B(t), C(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [6]. We must show that $\tau_e = \infty$ a.s. in order to demonstrate that this solution is global.

Assuming that $\zeta_0 \geq 1$ is sufficiently large, define the stop-time as follows for each integer $\zeta \geq \zeta_0$ for $P(0)$, $B(0)$ and $C(0)$ lying within the interval $\left[\frac{1}{\zeta_0}, \zeta_0\right]$:

$$\tau_\zeta = \inf\{t \in [0, \tau_e] : \min\{P(t), B(t), C(t)\} \leq \frac{1}{\zeta} \text{ or } \max\{P(t), B(t), C(t)\} \geq \zeta\}.$$

Set $\tau_\infty = \lim\limits_{\zeta\to\infty} \tau_\zeta$, where τ_ζ is growing as $\zeta \to \infty$ and $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set).

When $\tau_\infty \leq \tau_e$ a.s. We only need to show that $\tau_\infty = \infty$ to finish the proof if we can demonstrate that $\tau_\infty = \infty$ a.s., which means that $\tau_e = \infty$ and $(P(t), B(t), C(t)) \in \Omega$ for all $t \geq 0$.

If there is a violation of this statement, there exists a constant $T > 0$ and $\epsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \epsilon$.

Consequently, there is an integer $\zeta_1 \geq \zeta_0$ such that :

$$\mathbb{P}\{\tau_\zeta \leq T\} \geq \epsilon, \zeta \geq \zeta_1. \tag{5}$$

Define a C^2-function $\mathcal{V}:\mathbb{R}^3_+\to\mathbb{R}_+$ by

$$\mathcal{V}(P,B,C)=(P-1-logP)+(B-1-logB)+(C-1-logC).$$

Assume that $T>0$ and $\zeta\geq\zeta_0$ are arbitrary. Using Itô's formula, we arrive at

$$d\mathcal{V}(P,B,C)=\mathcal{L}\mathcal{V}dt+\sigma_1(P-1)dW_1(t)+\sigma_2(B-1)dW_2+\sigma_3(C-1)dW_3,$$

where $\mathcal{L}\mathcal{V}:\mathbb{R}^3_+\to\mathbb{R}_+$ is specified by

$$\begin{aligned}\mathcal{L}\mathcal{V}(P,B,C) &= \left(1-\frac{1}{P}\right)\left[A-k\mu\frac{PB}{A}-(\mu+\lambda)P\right]+\left(1-\frac{1}{B}\right)\left[k\mu\frac{PB}{A}-(\delta+\mu)B\right]\\ &\quad+\left(1-\frac{1}{C}\right)[\lambda P+\delta B-\mu C]\\ &= A-\mu P-\mu B-\mu C-A/P+k\mu B/A+3\mu+\lambda+\mu-k\mu P/A-\lambda P/C\\ &\quad-\delta B/C+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_3^2\\ &\leq A+3\mu+\lambda+\mu+k+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_3^2\\ &:= D.\end{aligned}$$

Next, we obtain

$$d\mathcal{V}\leq Ddt+\sigma_1(P-1)dW_1(t)+\sigma_2(B-1)dW_2+\sigma_3(C-1)dW_3.$$

Consequently, if $t_1\leq T$,

$$\int_0^{\tau_\zeta\wedge t_1}d\mathcal{V}(P(t),B(t),C(t))\leq\int_0^{\tau_\zeta\wedge t_1}Ddt+\int_0^{\tau_\zeta\wedge t_1}\sigma_1(P-1)dW_1(t)+\sigma_2(B-1)dW_2+\sigma_3(C-1)dW_3.$$

It follows from this that,

$$\begin{aligned}E[\mathcal{V}(P(\tau_\zeta\wedge t_1),B(\tau_\zeta\wedge t_1),C(\tau_\zeta\wedge t_1))] &\leq \mathcal{V}(P_0,B_0,C_0)+E\int_0^{\tau_\zeta\wedge t_1}Ddt\\ &\leq \mathcal{V}(P_0,B_0,C_0)+DT. \qquad (6)\end{aligned}$$

Set $\phi_\zeta=\{\tau_\zeta\leq T\}$ for $\zeta\geq\zeta_1$ and by (5), $\mathbb{P}(\phi_\zeta)\geq\epsilon$.
At least one of $P(\tau_\zeta,\phi)$, $B(\tau_\zeta,\phi)$ and $C(\tau_\zeta,\phi)$ equals either ζ or $\frac{1}{\zeta}$ for any $\phi\in\phi_\zeta$; therefore, $\mathcal{V}(P(\tau_\zeta,\phi),B(\tau_\zeta,\phi),C(\tau_\zeta,\phi))$ is no less than

$$\zeta-1-log\zeta \text{ or } \frac{1}{\zeta}-1-log\frac{1}{\zeta}=\frac{1}{\zeta}-1+log\zeta$$

Therefore

$$\mathcal{V}(P(\tau_\zeta,\phi),B(\tau_\zeta,\phi),C(\tau_\zeta,\phi))\geq min\{\zeta-1-log\zeta,\frac{1}{\zeta}-1+log\zeta\}.$$

Next, get from (5) and (6)

$$\begin{aligned}\mathcal{V}(P(0),B(0),C(0))+DT &\geq E\left[1_{\phi_\zeta}V(P(\tau_\zeta),B(\tau_\zeta),C(\tau_\zeta))\right]\\ &\geq \epsilon\min\{\zeta-1-log\zeta,\frac{1}{\zeta}-1+log\zeta\},\end{aligned}$$

in which ϕ_ζ's indicator function is 1_{ϕ_ζ}.
The contradiction $\infty > \mathcal{V}(P_0,B_0,C_0)+DT=\infty$ results from letting $\zeta\to\infty$. Therefore, $\tau_\infty=\infty$ a.s. must exist. □

3 Asymptotic Behavior of the boycott-free equilibrium E_0

For the stochastically perturbed system (3), theorem 1 shows that the global positive solution will persist. Asymptotic behavior analysis is essential because model (3) lacks an explicit solution. This theorem gives the asymptotic property of model (3).

Theorem 2. *If the following conditions are fulfilled and* $R_0\leq 1$*:*

$$\begin{aligned}\sigma_1^2 &< \frac{1}{b+2}\left[b\left(\mu+\frac{\lambda}{2}\right)+2(\mu+\lambda)\right],\\ \sigma_2^2 &< 2\left(\mu+\lambda-\frac{ac_2}{2}\right)-\frac{b}{2}\delta,\\ \sigma_3^2 &< \mu-\frac{\lambda+\delta}{2},\end{aligned}$$

therefore for any given initial value $(P(0),B(0),C(0))\in\Omega$*, the solution of model* (3) *has the property*

$$\lim_{t\to\infty} sup\frac{1}{t}E\int_0^t\left[\left(P(s)-\frac{A}{\lambda+\mu}\right)^2+B(s)^2+\left(C(s)-\frac{A\lambda}{\mu(\lambda+\mu)}\right)^2\right]ds\leq\frac{M}{L},$$

where

$$L=min\{b\left(\mu+\frac{\lambda}{2}\right)+2(\mu+\lambda)-(b+2)\sigma_1^2,2\left(\mu+\lambda-\frac{ac_2}{2}\right)-\frac{b}{2}\delta-\sigma_2^2,b\left(\mu-\frac{\lambda+\delta}{2}\right)-b\sigma_3^2\},$$

$M=(b+2)a^2\sigma_1^2+b\left(\frac{\lambda}{\mu}\right)^2a^2\sigma_3^2$*, and* b *is positive constant defined as* (9).

Proof. Let $u = P - a$, $v = B$, $w = C - \frac{\lambda}{\mu}a$, $a = \frac{A}{\lambda + \mu}$, then

$$\begin{cases} du(t) &= \left[-\frac{k\mu}{A}(u(t)+a)v(t) - (\mu+\lambda)u(t)\right]dt + \sigma_1(u(t)+a)dW_1(t), \\ dv(t) &= \left[\frac{k\mu}{A}(u(t)+a)v(t) - (\delta+\mu)v(t)\right]dt + \sigma_2 v(t)dW_2(t), \\ dw(t) &= [\lambda u(t) + \delta v(t) - \mu w(t)]\,dt + \sigma_3\left(w(t) + \frac{\lambda}{\mu}a\right)dW_3(t). \end{cases} \tag{7}$$

Define a C^2-function $\mathcal{V} : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$\mathcal{V}(u,v,w) = b\left(\frac{1}{2}u^2 + c_1 v + \frac{1}{2}w^2\right) + (u+v)^2 + c_2 v := b\mathcal{V}_1 + \mathcal{V}_2,$$

where

$$\mathcal{V}_1(u,v,w) = \frac{1}{2}u^2 + c_1 v + \frac{1}{2}w^2, \mathcal{V}_2(u,v,w) = (u+v)^2 + c_2 v.$$

The constants c_1 and c_2 are both positive. Using Itô's formula, we get

$$d\mathcal{V}_1(u,v,w) = \mathcal{L}\mathcal{V}_1 dt + \sigma_1 u(u+a)dW_1(t) + \sigma_2 c_1 v dW_2(t) + \sigma_3 w\left(w + \frac{\lambda}{\mu}a\right)dW_3(t),$$

where

$$\begin{aligned} \mathcal{L}\mathcal{V}_1 &= -\frac{k\mu}{A}(u+a)uv - (\mu+\lambda)u^2 + c_1\frac{k\mu}{A}(u+a)v - c_1(\delta+\mu)v \\ &\quad + w\lambda(u+a) + w(\delta v - \mu w - \lambda a) + \frac{1}{2}\sigma_1^2(u+a)^2 + \frac{1}{2}\sigma_3^2(w + \frac{\lambda}{\mu}a)^2 \\ &= -(\mu+\lambda)u^2 - \frac{k\mu}{A}u^2 v - \frac{k\mu}{A}uv(a - c_1) - c_1\frac{k\mu}{A}a\left(\frac{1}{R_0} - 1\right)v - \mu w^2 \\ &\quad + \lambda uw + \delta vw + \frac{1}{2}\sigma_1^2(u+a)^2 + \frac{1}{2}\sigma_3^2(w + \frac{\lambda}{\mu}a)^2, \end{aligned}$$

$R_0 \leq 1$ is utilized. In this case, we select $c_1 = a$ so that $a - c_1 = 0$.

$$\begin{aligned} \mathcal{L}\mathcal{V}_1 &\leq -(\mu+\lambda)u^2 - \mu w^2 + \frac{\lambda}{2}(u^2 + w^2) + \frac{\delta}{2}(v^2 + w^2) + \frac{1}{2}\sigma_1^2(u+a)^2 \\ &\quad + \frac{1}{2}\sigma_3^2(w + \frac{\lambda}{\mu}a)^2 \\ &\leq -(\mu + \frac{\lambda}{2})u^2 + \frac{1}{2}\delta v^2 - \frac{1}{2}(2\mu - \lambda - \delta)w^2 + \sigma_1^2 u^2 + \sigma_1^2 a^2 + \sigma_3^2 w^2 + \sigma_3^2\left(\frac{\lambda}{\mu}a\right)^2. \end{aligned}$$

In the same way, when we use Itô's formula to $\mathcal{V}_2(u,v,w)$, we obtain

$$d\mathcal{V}_2(u,v,w) = \mathcal{L}\mathcal{V}_2 dt + (2u+2v)\sigma_1(u+a)dW_1(t) + (2u+2v+c_2)\sigma_2 v dW_2(t),$$

where

$$\begin{aligned}
\mathcal{L}\mathcal{V}_2 &= (2u+2v)\left[-\frac{k\mu}{A}(u+a)v-(\mu+\lambda)u\right]+(2u+2v+c_2)\left[\frac{k\mu}{A}(u+a)v-(\delta+\mu)v\right]\\
&\quad +\sigma_1^2(u+a)^2+\sigma_2^2v^2\\
&= (2u+2v)\left[-(\mu+\lambda)u-(\delta+\mu)v\right]+c_2\left[\frac{k\mu}{A}(u+a)v-(\delta+\mu)v\right]\\
&\quad +\sigma_1^2(u+a)^2+\sigma_2^2v^2\\
&= -2(\mu+\lambda)u^2-2(\mu+\delta)v^2+uv\left(-2(\mu+\lambda)-2(\mu+\delta)+c_2k\mu/A\right)+ac_2v\\
&\quad -c_2(\delta+\mu)v+\sigma_1^2(u+a)^2+\sigma_2^2v^2.
\end{aligned}$$

Here, $c_2 = \dfrac{2A}{\mu k}(2\mu+\lambda+\delta)$ is selected so that $-2(\mu+\lambda)-2(\mu+\delta)+\dfrac{c_2k\mu}{A}=0$

$$\begin{aligned}
\mathcal{L}\mathcal{V}_2 &\leq -2(\mu+\lambda)u^2-2(\mu+\delta)v^2+ac_2v+\sigma_1^2(u+a)^2+\sigma_2^2v^2\\
&\leq -2(\mu+\lambda)u^2-2(\mu+\delta-\frac{ac_2}{2})v^2+2\sigma_1^2u^2+\sigma_2^2v^2+2\sigma_1^2a^2.
\end{aligned}$$

Consequently, we can acquire

$$\begin{aligned}
\mathcal{L}\mathcal{V}(u,v,w) &= b\mathcal{V}_1+\mathcal{V}_2\\
&\leq -\left[b\left(\mu+\frac{\lambda}{2}\right)+2(\mu+\lambda)-(b+2)\sigma_1^2\right]u^2-\left[2\left(\mu+\lambda-\frac{ac_2}{2}\right)-\frac{b}{2}\delta-\sigma_2^2\right]v^2\\
&\qquad -\left[b\left(\mu-\frac{\lambda+\delta}{2}\right)-b\sigma_3^2\right]w^2+M, \qquad (8)
\end{aligned}$$

where $M=(b+2)a^2\sigma_1^2+b\left(\dfrac{\lambda}{\mu}\right)^2a^2\sigma_3^2$. Let us choose

$$b > max\left\{\frac{2}{6\mu+5\lambda},\frac{\delta}{2(2\mu+2\lambda-ac_2)},\frac{2}{2\mu-\lambda-\delta}\right\}, \qquad (9)$$

such that

$$2(\mu+\delta)-\left(ac_2+\frac{\delta}{2}\right)>0.$$

Establish

$$L = min\{b\left(\mu+\frac{\lambda}{2}\right)+2(\mu+\lambda)-(b+2)\sigma_1^2, 2\left(\mu+\lambda-\frac{ac_2}{2}\right)-\frac{b}{2}\delta-\sigma_2^2,$$

$$b\left(\mu - \frac{\lambda+\delta}{2}\right) - b\sigma_3^2\},$$

putting this into (8) results in

$$d\mathcal{V} \le (-Lu^2 - Lv^2 - Lw^2 + M)dt + \sigma_1\left[(b+2)u + 2v\right](u+a)dW_1(t)$$
$$+\sigma_2(bc_1 + 2u + 2v + c_2)v dW_2(t) + \sigma_3 wb\left(w + \frac{\lambda}{\mu}a\right)dW_3(t).$$

Taking the expectation and integrating this from 0 to t, results in

$$E\mathcal{V}(t) - \mathcal{V}(0) \le -E\int_0^t (Lu^2 + Lv^2 + Lw^2 + M)ds.$$

Therefore,

$$\lim_{t\to\infty} sup \frac{1}{t} E\int_0^t \left[u^2(s) + v^2(s) + w^2(s)\right] ds \le \frac{M}{L}.$$

As a result,

$$\lim_{t\to\infty} sup \frac{1}{t} E\int_0^t \left[\left(P(s) - \frac{A}{\lambda+\mu}\right)^2 + B^2(s) + \left(C(s) - \frac{\lambda}{\mu}\frac{A}{\lambda+\mu}\right)^2\right] ds \le \frac{M}{L}.$$

This completes Theorem 2's proof. □

Remark 1. *Theorem 2 demonstrates that, under certain conditions, the solution of system (3) will oscillate around the deterministic model's boycott-free equilibrium and that the intensity of the disturbance is proportionate to the white noise's intensity. In a economical perspective, as the intensity of stochastic perturbations is small, we consider the boycott will stop.*
Furthermore, E_0 is the boycott-free equilibrium of system (3) if $\sigma_1 = 0$ and $\sigma_3 = 0$. From Theorem 2's proof, we obtain

$$\mathcal{L}\mathcal{V}(u,v,w) \le -\left[b\left(\mu + \frac{\lambda}{2}\right) + 2(\mu+\lambda) - (b+2)\sigma_1^2\right]u^2$$
$$-\left[2\left(\mu + \lambda - \frac{ac_2}{2}\right) - \frac{b}{2}\delta - \sigma_2^2\right]v^2 - \left[b\left(\mu - \frac{\lambda+\delta}{2}\right) - b\sigma_3^2\right]w^2.$$

Therefore, system (3)'s solution is asymptotically stochastically stable in the large.

4 Ergodicity of System (3)

We also want to know when a populace will start to boycott a product and stay that way. The problem in the deterministic models is resolved by demonstrating that the appropriate model's boycotted equilibrium either has a global attractor or is globally asymptotically stable. However, the stochastic system does not have a boycotted equilibrium. Based on the theory of Hasminskii [2], this part we demonstrate the existence of a stationary distribution on R_0, subject to certain conditions. Considering the model's parameters, which indicate that the boycott will also succeed and prevail. We introduce some stationary distribution theory here (see Hasminskii [2]). Given $X(t)$ is a homogeneous Markov process in E_l (where E_l indicates Euclidean l-space), let $X(t)$ be characterized by the stochastic differential equation as follows:

$$dX(t) = h(X)dt + \sum_{s=1}^{k} g_s(X)dW_s(t). \tag{10}$$

The diffusion matrix is $A(x) = (a_{ij}(x))$, $a_{ij}(x) = \sum_{s=1}^{k} g_s^i(x)g_s^j(x)$

Assumption 1: There exists a bounded domain $U \subset E_l$ with regular boundary Γ, having the following properties that
$(A1)$ In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.
$(A2)$ If $x \in E_l - U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup\limits_{x\in K} E_x\tau < +\infty$ for every compact subset $K \subset E_l$.

Lemma 1. *(Hasminskii [2]) If Assumption 1 holds, then the Markov process $X(t)$ has a stationary distribution $\mu(.)$. Let $f(.)$ be a function integrable with respect to the measure μ. Then*

$$\mathbb{P}_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{E_l} f(x)\mu(dx)dt\right\} = 1, \forall x \in E_l$$

Remark 2. *To validate $(A1)$, it is sufficient to show that F is uniformly elliptical in D, where $F_u = b(x).u_x + 0.5tr(A(x)u_{xx})$, that is, there is a positive number Z such that*

$$\sum_{i,j=1}^{k} a_{ij}(x)\xi_i\xi_j \geq Z|\xi|^2, x \in \overline{D}, \xi \in \mathbb{R}^k.$$

(see [[3], P.103] and Rayleigh's principale in [[4], P.349]). To validate $(A2)$, it is sufficent to prove that there is neighborhood D and non-negative C^2-function such that for any $x \in E_l - U$, $\mathcal{LV}(x)$ is negative (see [[5], p.1163]).

Lemma 2. *A regular, temporally homogeneous Markov process in E_l is denoted by $X(t)$. $X(t)$ is recurrent relative to any nonempty domain in E_l if it is recurrent relative to some bounded domain U.*
It is possible to express system (3) as a variant of System (10):

$$d\begin{pmatrix} P \\ B \\ C \end{pmatrix} = \begin{pmatrix} A - \frac{k\mu}{A}PB - (\mu+\lambda)P \\ \frac{k\mu}{A}PB - (\delta+\mu)B \\ \lambda P + \delta B - \mu C \end{pmatrix} dt + \begin{pmatrix} \sigma_1 P \\ 0 \\ 0 \end{pmatrix} dW_1(t)$$

$$+\begin{pmatrix}0\\ \sigma_2 B\\ 0\end{pmatrix} dW_2(t)+\begin{pmatrix}0\\ 0\\ \sigma_3 C\end{pmatrix} dW_3(t)$$

and the diffusion matrix is $A = diag(\sigma_1^2 P^2, \sigma_2^2 B^2, \sigma_3^2 C^2)$.

Theorem 3. *If the following conditions are fulfilled and* $R_0 > 1$*:*

$$\begin{aligned}\sigma_1^2 &< \frac{1}{n+2}\left[n\left(\mu+\frac{\lambda}{2}\right)+2(\mu+\lambda)\right],\\ \sigma_2^2 &< \mu+\left(1-\frac{n}{4}\right)\delta,\\ \sigma_3^2 &< \mu-\frac{\lambda+\delta}{2},\end{aligned}$$

such that

$$S < min\{\left[n\left(\mu+\frac{\lambda}{2}\right)+2(\mu+\lambda)-(n+2)\sigma_1^2\right](P^*)^2, \left[2n(\mu+\delta)-\frac{\delta}{2}-2\sigma_2^2\right](B^*)^2,$$

$$\left[n\left(\mu-\frac{\lambda+\delta}{2}\right)-n\sigma_3^2\right](C^*)^2\}. \tag{11}$$

System (3) *then has an ergodic property and a staionary distribution* $\mu(.)$ *for any initial value* $(P(0), B(0), C(0)) \in \Omega$*, where* $S = (n+2)\sigma_1^2(P^*)^2 + 2\sigma_2^2(B^*)^2 + n\sigma_3^2(C^*)^2$*, and* n *is the positive constant specified as* (12)*.*

Proof. Define a C^2-function $\mathcal{V} : \mathbb{R}_+^3 \to \mathbb{R}_+$ by

$$\begin{aligned}\mathcal{V}(P,B,C) &= n\left[\frac{1}{2}(P-P^*)^2 + a_1\left(B-B^*-B^*log\frac{B}{B^*}\right)+\frac{1}{2}(C-C^*)^2\right]\\ &\quad +[(P-P^*)+(B-B^*)]^2 + a_2\left(B-B^*-B^*log\frac{B}{B^*}\right)\\ &\quad := n\mathcal{V}_1+\mathcal{V}_2,\end{aligned}$$

where

$$\begin{aligned}\mathcal{V}_1(P,B,C) &= \frac{1}{2}(P-P^*)^2 + a_1\left(B-B^*-B^*log\frac{B}{B^*}\right)+\frac{1}{2}(C-C^*)^2\\ \mathcal{V}_2(P,B,C) &= [(P-P^*)+(B-B^*)]^2 + a_2\left(B-B^*-B^*log\frac{B}{B^*}\right),\end{aligned}$$

The constants n, a_1, and a_2 are all positive. Using Itô's formula, we get

$$d\mathcal{V}_1 = \mathcal{L}\mathcal{V}_1 dt + \sigma_1 P(P-P^*)dW_1(t) + a_1\sigma_2(B-B^*)dW_2(t) + \sigma_3 C(C-C^*)dW_3(t),$$

where

$$
\begin{aligned}
\mathcal{L}\mathcal{V}_1 &= (P-P^*)\left[A-\frac{k\mu}{A}PB-(\lambda+\mu)P\right]+\frac{1}{2}\sigma_1^2P^2+a_1(B-B^*)\left[\frac{k\mu}{A}P-(\delta+\mu)\right]\\
&\quad+(C-C^*)\left[\lambda P+\delta B-\mu C\right]+\frac{1}{2}\sigma_3^2C^2\\
&= (P-P^*)\left[-\frac{k\mu}{A}(P-P^*)B-\frac{k\mu}{A}P^*(B-B^*)-(\mu+\lambda)(P-P^*)\right]\\
&\quad+a_1\frac{k\mu}{A}(B-B^*)(P-P^*)+(C-C^*)\left[\lambda(P-P^*)+\delta(B-B^*)-\mu(C-C^*)\right]\\
&\quad+\frac{1}{2}\sigma_1^2P^2+\frac{1}{2}\sigma_3^2C^2\\
&= -(\mu+\lambda)(P-P^*)^2-\frac{k\mu}{A}(P-P^*)^2B-\mu(C-C^*)^2\\
&\quad+(B-B^*)(P-P^*)\left(\frac{a_1k\mu}{A}-\frac{k\mu}{A}P^*\right)+(C-C^*)[\lambda(P-P^*)+\delta(B-B^*)]\\
&\quad+\frac{1}{2}\sigma_1^2\left[(P-P^*)+P^*\right]^2+\frac{1}{2}\sigma_3^2\left[(C-C^*)+C^*\right]^2.
\end{aligned}
$$

Here, $a_1 = P^*$ is selected so that $\dfrac{a_1k\mu}{A}-\dfrac{k\mu}{A}P^*=0$,

$$
\begin{aligned}
\mathcal{L}\mathcal{V}_1 &\le -(\lambda+\mu)(P-P^*)^2-\mu(C-C^*)^2+\frac{1}{2}\lambda(C-C^*)^2+\frac{1}{2}\lambda(P-P^*)^2+\frac{1}{2}\delta(C-C^*)^2\\
&\quad+\frac{1}{2}\delta(B-B^*)^2+\frac{1}{2}\sigma_1^2\left[(P-P^*)+P^*\right]^2+\frac{1}{2}\sigma_3^2\left[(C-C^*)+C^*\right]^2\\
&\le -\left(\mu+\frac{\lambda}{2}\right)(P-P^*)^2+\frac{\delta}{2}(B-B^*)^2-\left(\mu-\frac{\lambda+\delta}{2}\right)(C-C^*)^2+\sigma_1^2(P-P^*)^2\\
&\quad+\sigma_1^2(P^*)^2+\sigma_3^2(C-C^*)^2+\sigma_3^2(C^*)^2.
\end{aligned}
$$

In the same way, when we use Itô's formula to $\mathcal{V}_2(P,B,C)$, we obtain

$$
\begin{aligned}
d\mathcal{V}_2 &= \mathcal{L}\mathcal{V}_2dt+[2(P-P^*)+2(B-B^*)]\sigma_1PdW_1(t)\\
&\quad+\left[2(P-P^*)+2(B-B^*)+a_2\left(1-\frac{B^*}{B}\right)\right]\sigma_2BdW_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}\mathcal{V}_2 &= [2(P-P^*)+2(B-B^*)][A-(\lambda+\mu)P-(\delta+\mu)B]\\
&\quad+a_2(B-B^*)\left[\frac{k\mu}{A}P-(\delta+\mu)\right]+\sigma_1^2P^2+\sigma_2^2B^2\\
&= [2(P-P^*)+2(B-B^*)][-(\lambda+\mu)(P-P^*)-(\delta+\mu)(B-B^*)]\\
&\quad+a_2\frac{k\mu}{A}(B-B^*)(P-P^*)+\sigma_1^2[(P-P^*)+P^*]^2+\sigma_2^2[(B-B^*)+B^*]^2
\end{aligned}
$$

$$= -2(\lambda+\mu)(P-P^*)^2 - 2(\delta+\mu)(B-B^*)^2$$
$$+(P-P^*)(B-B^*)\left[a_2\frac{k\mu}{A} - 2(\lambda+\mu) - 2(\delta+\mu)\right] + \sigma_1^2[(P-P^*)+P^*]^2$$
$$+\sigma_2^2[(B-B^*)+B^*]^2,$$

Here we take $a_2 = \frac{2A}{\mu k}(2\mu+\lambda+\delta)$ such that $-2(\mu+\lambda) - 2(\mu+\delta) + \frac{a_2 k\mu}{A} = 0$,

$$\begin{aligned}\mathcal{LV}_2 &\leq -2(\lambda+\mu)(P-P^*)^2 - 2(\delta+\mu)(B-B^*)^2 + \sigma_1^2[(P-P^*)+P^*]^2 \\ &\quad +\sigma_2^2[(B-B^*)+B^*]^2 \\ &\leq -2(\lambda+\mu)(P-P^*)^2 - 2(\delta+\mu)(B-B^*)^2 + 2\sigma_1^2(P-P^*)^2 + 2\sigma_1^2(P^*)^2 \\ &\quad +2\sigma_2^2(B-B^*)^2 + 2\sigma_2^2(B^*)^2.\end{aligned}$$

Consequently, we can acquire

$$\begin{aligned}\mathcal{LV}(u,v,w) &= n\mathcal{V}_1 + \mathcal{V}_2 \\ &\leq -\left[n\left(\mu+\frac{\lambda}{2}\right) + 2(\mu+\lambda) - (n+2)\sigma_1^2\right](P-P^*)^2 \\ &\quad -\left[2(\mu+\delta) - \frac{n}{2}\delta - 2\sigma_2^2\right](B-B^*)^2 - \left[n\left(\mu - \frac{\lambda+\delta}{2}\right) - n\sigma_3^2\right](C-C^*)^2 + S,\end{aligned}$$

where $S = (n+2)\sigma_1^2(P^*)^2 + 2\sigma_2^2(B^*)^2 + n\sigma_3^2(C^*)^2$. Let us choose

$$n > max\left\{\frac{2}{6\mu+5\lambda}, \frac{\delta}{4(\delta+\lambda)}, \frac{2}{2\mu-\lambda-\delta}\right\}, \tag{12}$$

such that

$$\frac{3}{2}\delta + 2\mu > 0.$$

Because of (11), the ellipsoid

$$-\left[n\left(\mu+\frac{\lambda}{2}\right) + 2(\mu+\lambda) - (n+2)\sigma_1^2\right](P-P^*)^2 - \left[2n(\mu+\delta) - \frac{\delta}{2} - 2\sigma_2^2\right](B-B^*)^2$$
$$-\left[n\left(\mu - \frac{\lambda+\delta}{2}\right) - n\sigma_3^2\right](C-C^*)^2 + S = 0,$$

lies entirely in $\mathbb{R}_+^3$.

Given that $\overline{U} \subseteq E_l = \mathbb{R}_+^3$, we can assume that U is a neighborhood of the ellipsoid. Accordingly, for $x \in \mathbb{R}_+^3 - U$, $\mathcal{LV} \leq -Z$ (where Z is a positive constant), indicating that condition (A.2) in lemma 1 is satisfied.
Therefore, the solution $X(t) = (P(t), B(t), C(t))$ is recurrent in the domain U, which, together with lemma 2 and remark 2, implies that $X(t)$ is recurrent in any bounded

domain $D \subset \mathbb{R}^3_+$. Besides, $\forall D$, there is a $Z = min\{\sigma_1^2 P^2, \sigma_2^2 B^2, \sigma_3^2 C^2\} > 0$, such that

$$\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j = \sigma_1^2 P^2 \xi_1^2 + \sigma_2^2 B^2 \xi_2^2 + \sigma_3^2 C^2 \xi_3^2 \geq Z\|\xi\|^2,$$

for all $X \in \overline{D}$, $\xi \in \mathbb{R}^3$, which implies that the condition (A.1) is likewise met. Consequently, the stochastic system (3) is ergodic and has a stationary distribution $\mu(.)$. □

5 Numerical Simulations

In this section, we provide numerical simulations of the stochastic model (3) in order to validate the theoretical results presented in this study. The Matlab code was written and compiled using the Euler-Maruyama Method [26]. This simulation's parameters are derived from our basic work [10], which offers the system (1)'s stability analysis numerical simulations of the Moroccans' boycott of Centrale Danone [16].

First, we aim to test theorem 2, which states that the solution of system (3) will oscillate around the deterministic model's boycott-free equilibrium E_0. In figure (1), the parameters choosen are $A = 1375244$, $\lambda = 0.6$, $\delta = 0.01$, $\mu = 0.053$, $k = 0.4$, and $\sigma_1 = \sigma_2 = \sigma_3 = 0.02$. We noticed that in this work, the model units for the parameters are in days.

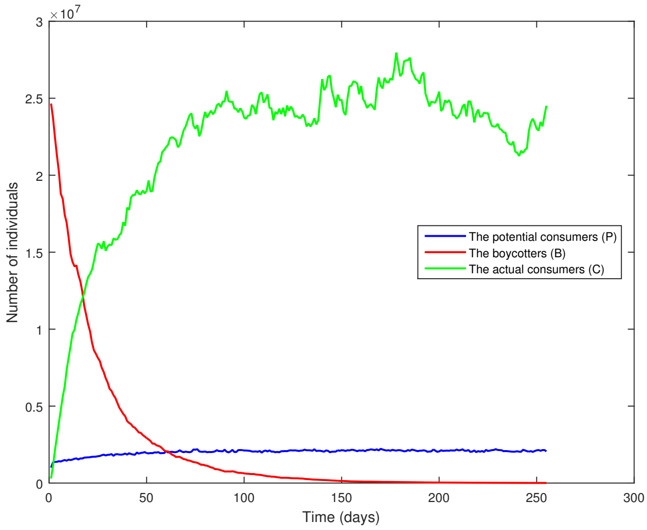
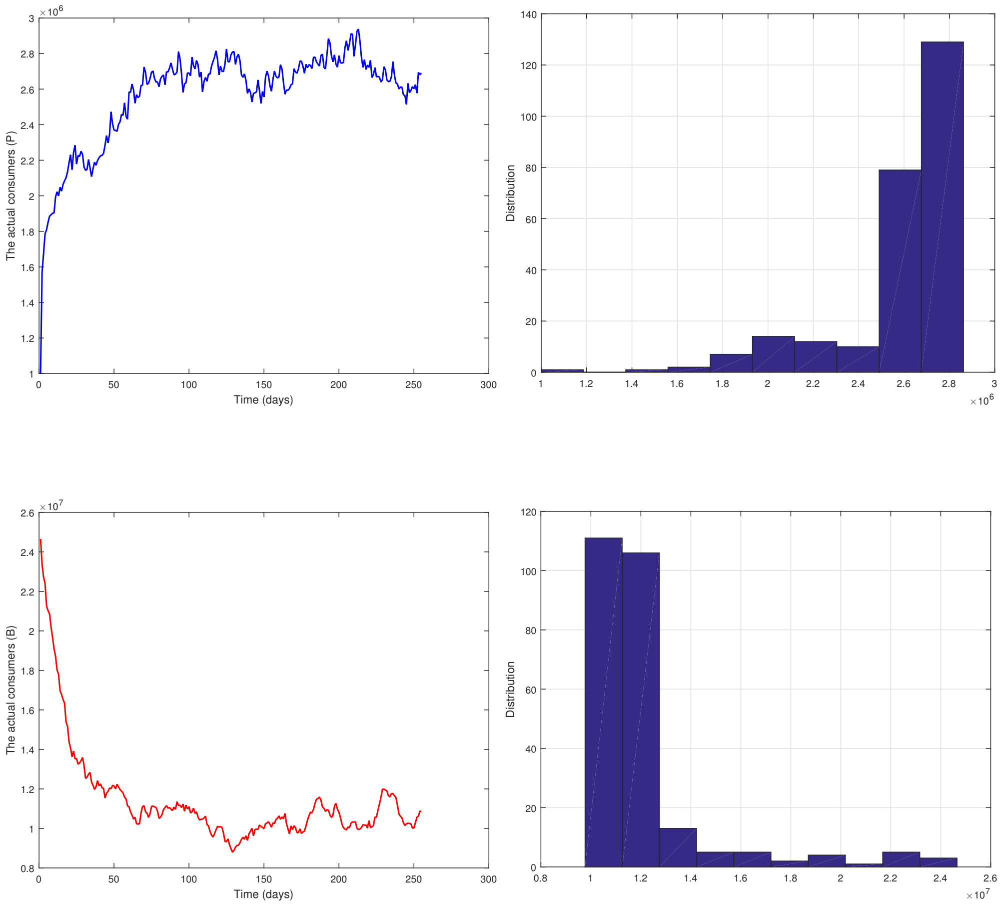


Fig. 1 The convergence of the solution oscillates around the boycott free equilibruim $E_0 = (2.106 \times 10^6, 0, 2.384 \times 10^7)$.

Secondly, is about theorem 3 that claims there exists an ergodic stationary distribution of stochastic model (3). The simulation results can be seen in figure 2, which clearly supports these results that the boycott of Centrale Danone would succed and prevail among population as a result of several reasens [16]. The parameters choosen are $A = 1375244$, $\lambda = 0.2$, $\delta = 0.01$, $\mu = 0.053$, $k = 0.6$, $\sigma_1 = 0.015$, $\sigma_2 = 0.02$, and $\sigma_3 = 0.025$.



6 Conclusion

In this paper, we studied the stochastic behavior of the boycotted commercial product model. We found that the system (3) has a unique global positive solution. We do an extensive investigation of the stochastic model's dynamic behavior with respect to the basic reproduction number, R_0.

If $R_0 \leq 1$, the model's solution oscillates around a steady state, which is the corresponding deterministic model's boycotted-free equilibrium. The boycott will succeed

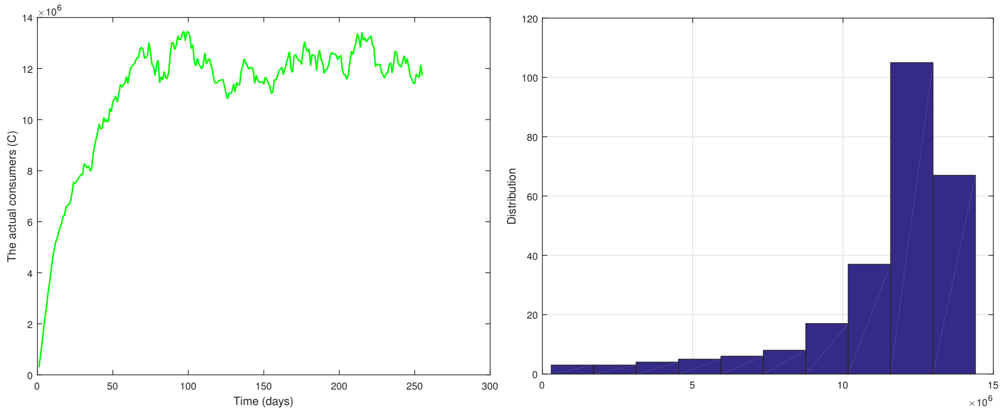


Fig. 2 The stochastic model (3) has an ergodic property; the population size with time (days) is depicted in the images on the left and right.

and prevail among consumers if $R_0 > 1$ because there is a stationary distribution and the solution has the ergodic property. We will study this model in both space and time, as well as using stochastic outcomes to add an optimal control problem to it.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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