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Trung-Thuy Kieu, Duc-Trong Luong, and Hoang-Long Ngo

Hanoi National University of Education, Cau Giay, Hanoi, Vietnam

ABSTRACT
We propose a tamed-adaptive Euler-Maruyama approximation scheme for stochastic differential equations with locally Lipschitz continuous, polynomial growth drift, and locally Hölder continuous, polynomial growth diffusion coefficients. We consider the strong convergence and the stability of the new scheme. In particular, we show that under some sufficient conditions for the stability of the exact solution, the tamed-adaptive scheme converges strongly in both finite and infinite time intervals.

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1. Introduction

Throughout this paper, we consider the process $(X_t)_{t\geq 0}$ given by the following stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad x_0 \in \mathbb{R}, t \geq 0, \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions; b and σ are real-valued measurable functions. Such SDEs arise in many areas of science and engineering, from population genetics to financial mathematics (see, e.g., Chapter 7 in [1], and [2]). Therefore, it is important to find effective methods to solve SDEs numerically. If coefficients b and σ are Lipschitz continuous, the numerical approximation for X has been well studied (see [1, 3]). However, numerical analysis for SDEs with non-Lipschitz coefficients is still a very active research area. In [4], Hutzenthaler et al. showed the divergence of the classical Euler-Maruyama approximation in L^p-norm when applying for some classes of SDEs with super-linear growth coefficients. After that, they introduced in [5] a tamed Euler-Maruyama approximation scheme and showed its strong convergence when applying for SDEs with locally Lipschitz continuous, super-linear growth drift coefficient and globally Lipschitz continuous diffusion coefficient. Since then, the tamed Euler-Maruyama has been developed by many authors for larger classes

CONTACT Hoang-Long Ngo ngolong@hnue.edu.vn Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam.

of SDEs with super-linear growth and low regularity coefficients (see, e.g., [6–10] and the references therein).

The strong convergence in infinite time of approximated solutions has attracted much attention recently. In [11], Fang and Giles introduced an adaptive Euler-Maruyama approximation. They showed that if b and σ satisfy the contractive Lipschitz condition (Assumption 9 in [11]), b is polynomial growth Lipschitz continuous, and σ is globally Lipschitz continuous, then the adaptive Euler-Maruyama approximation converges in L^p-norm in infinite time intervals. In [12], Li et al. studied the strong convergence in infinite time of a truncated Euler-Maruyama approximation scheme for SDEs with locally Lipschitz and polynomial growth coefficients. Their method is to use Khasminski's techniques on some well-designed Lyapunov functions.

The aim of this paper is to propose a numerical scheme that strongly converges in both finite and infinite time intervals for some class of one-dimensional SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients. By combining the tamed Euler-Maruyama approximation in [5, 9, 10] with the adaptive one in [11], we propose a tamed-adaptive Euler-Maruyama approximation scheme. We study the strong convergence of the scheme in both finite and infinite time intervals. The convergence rate to be established is optimal because it is similar to the standard results of the classical and tamed Euler-Maruyama schemes for SDEs with Hölder continuous diffusion coefficient (see [7, 8, 13, 14]). We also consider the stability of the tamed-adaptive scheme in L^p-norm where we obtain a similar result to Theorem 5 in [11] for a more general class of diffusion coefficients. The key idea of our argument is to use the Yamada-Watanabe approximation to obtain upper bounds for some p-th moments of the approximated solution. In comparison to [12], our method works for SDEs with locally Hölder continuous diffusion coefficient. However, a disadvantage point of using the Yamada-Watanabe approximation is that it can be applied only for one-dimensional SDEs while the method of Lyapunov functions in [11, 12] works for multi-dimensional SDEs.

The rest of this paper is organized as follows. The definition of tamed-adaptive Euler-Maruyama scheme and its convergence are established in Section 2. All the proofs are deferred to Section 3.

2. Main results

2.1. Assumptions

We consider the following assumptions on b and σ.

A1. There exist $\gamma \in \mathbb{R}, \eta \in [0, +\infty)$, and $p_0 \in [2, +\infty)$ such that

$$xb(x) + \frac{p_0 - 1}{2}|\sigma(x)|^2 \leq \gamma|x|^2 + \eta,$$

for any $x \in \mathbb{R}$.

A2. b is one-sided Lipschitz: there exists a constant L_1 such that

$$(x-y)(b(x)-b(y)) \leq L_1|x-y|^2,$$

for any $x, y \in \mathbb{R}$.

A3. b is locally Lipschitz continuous: there exist positive constants l and L_2 such that

$$|b(x)-b(y)| \leq L_2\left(1+|x|^l+|y|^l\right)|x-y|,$$

for any $x, y \in \mathbb{R}$.

A4. σ is $\left(\alpha+\frac{1}{2}\right)$-locally Hölder continuous: there exist positive constants m, L_3 and $\alpha \in [0, \frac{1}{2}]$ such that

$$|\sigma(x)-\sigma(y)| \leq L_3\left(1+|x|^m+|y|^m\right)|x-y|^{1/2+\alpha},$$

for any $x, y \in \mathbb{R}$.

It follows from Theorem 2.1 in [8] that if Assumptions **A1**, **A3**, **A4** hold for $p_0 \geq 4l+4$, then Equation (1) has a unique strong solution. It is straighforward to verify that the result of that Theorem 2.1 also holds for $\alpha = 0$.

2.2. Tamed-adaptive Euler-Maruyama scheme

For each $\Delta \in (0,1)$, the tamed-adaptive Euler-Maruyama discretization of Equation (1) is defined as follows

$$\begin{cases} t_0 = 0, \quad \hat{X}_0 = x_0, \quad t_{k+1} = t_k + h_\Delta(\hat{X}_{t_k}), \\ \hat{X}_{t_{k+1}} = \hat{X}_{t_k} + b_\Delta(\hat{X}_{t_k})h_\Delta(\hat{X}_{t_k}) + \sigma_\Delta(\hat{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \end{cases} \tag{2}$$

where

$$h_\Delta(x) = \frac{\Delta}{(1+|b(x)|+|\sigma(x)|+|x|^l)^2}, \tag{3}$$

for some constant $l \geq 1$, and b_Δ, σ_Δ are some approximations of b and σ which will be specified later.

The next result provides a sufficient condition for $t_k \to \infty$ as $k \to \infty$, which implies that the tamed adaptive approximation scheme (2) is well-defined.

Proposition 2.1. *Suppose that there exist positive constants L and β such that coefficients b, σ, b_Δ and σ_Δ satisfy the following conditions*

T1. $|b(x)| \vee |\sigma(x)| \leq L(1+|x|^\beta)$;
T2. $x(b_\Delta(x) - b_\Delta(0)) \leq L|x|^2$;
T3. $|\sigma_\Delta(x)| \leq L|\sigma(x)|$ *and* $|b_\Delta(x)| \leq L|b(x)|$;
T4. $|\sigma_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$;

for any $x \in \mathbb{R}$. Then

$$\lim_{k\to+\infty} t_k = +\infty \quad \text{a.s.} \tag{4}$$

The proof is deferred to Section 3.3.

Under the assumptions of Proposition 2.1, we can define $\underline{t} := \max\{t_n : t_n \le t\}$ for the nearest time point before t, and $N_t := \max\{n : t_n \le t\}$ for the number of time steps approximation up to time t. Note that $\underline{t}$ is a stopping time. We define the piecewise constant interpolant process $\bar{X}_t = \hat{X}_{\underline{t}}$ and also define the standard continuous interpolant as

$$\begin{aligned}\hat{X}_t &= \hat{X}_{\underline{t}} + b_\Delta\left(\hat{X}_{\underline{t}}\right)(t-\underline{t}) + \sigma_\Delta\left(\hat{X}_{\underline{t}}\right)\left(W_t - W_{\underline{t}}\right)\\ &= \bar{X}_t + b_\Delta(\bar{X}_t)(t-\underline{t}) + \sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}}).\end{aligned} \tag{5}$$

Hence, $\hat{X}_t$ is the solution of the SDE

$$d\hat{X}_t = b_\Delta(\bar{X}_t)dt + \sigma_\Delta(\bar{X}_t)dW_t, \qquad \hat{X}_0 = x_0.$$

Remark 2.2. It is straightforward to verify that under Assumptions **A1–A4**, the following functions

$$b_\Delta(x) = b(x), \quad \sigma_\Delta(x) = \frac{\sigma(x)}{1+\Delta^{1/2}|\sigma(x)|}. \tag{6}$$

satisfy all conditions of Proposition 2.1.

2.3. Moments

We first consider the moments of the exact solution. The following result should be known but we are not able to find it in any classical texts. The proof is given in Section 3.1.

Proposition 2.3. *Assume that coefficients b and σ satisfy the condition **A1**, and σ is bounded on every compact subset of $\mathbb{R}$. Then, for any $p \in [0, p_0]$,*

$$\mathbb{E}\left[|X_t|^p\right] \le \begin{cases} |x_0^2 e^{2\gamma t} + \frac{\eta}{\gamma}(e^{2\gamma t}-1)|^{p/2} & \text{if } \gamma \ne 0,\\ |x_0^2 + 2\eta t|^{p/2} & \text{if } \gamma = 0.\end{cases} \tag{7}$$

We have the following bound on moments of the tamed-adaptive Euler-Maruyama approximation.

Theorem 2.4. *If coefficients $b, \sigma, b_\Delta, \sigma_\Delta$ satisfy Conditions **T1–T4**, and there exist $\gamma \in \mathbb{R}, \eta \in [0, +\infty]$ and $p_0 \in [2, +\infty)$ such that for all $x \in \mathbb{R}$ and $t \ge 0$, it holds*

$$xb_\Delta(x) + \frac{p_0 - 1}{2}\sigma_\Delta^2(x) \le \gamma|x|^2 + \eta. \tag{8}$$

Then, for any positive integer $k \le p_0/2$, there exists a positive constant $K = K(x_0, k, \eta, \gamma, L)$, which does not depend on t or Δ, such that

$$\mathbb{E}\left[|\hat{X}_t|^{2k}\right]\vee\mathbb{E}\left[|\bar{X}_t|^{2k}\right] \leq \begin{cases} Ke^{2k\gamma t} & \text{if } \gamma > 0 \\ K(1+t)^k & \text{if } \gamma = 0 \\ K & \text{if } \gamma < 0 \end{cases}. \tag{9}$$

The proof of Theorem 2.4 is deferred to Section 3.4.

Remark 2.5. Under Assumptions **A1–A4**, functions b_Δ and σ_Δ defined in (6) satisfy conditions **T2–T4** and (8). If $\gamma < 0$, then the approximated solution is stable in the sense that for any $0 \leq p \leq 2[p_0/2]$ there exits a constant K, which does not depend on Δ, such that

$$\sup_{t\geq 0} \mathbb{E}\left[|\hat{X}_t|^p\right]\vee\mathbb{E}\left[|\bar{X}_t|^p\right] < K.$$

Here we use the notation $[p_0/2]$ for the integer part of $p_0/2$. Therefore, our result slightly improves Theorem 5 in [11] since we are able to relax the boundedness condition on σ. This improvement is due to the Yamada-Watanabe approximation.

Remark 2.6. Let N_T be the number of timesteps required by a path approximation on $[0, T]$ for any $T > 0$. Suppose that all conditions of Theorem 2.4 hold, then we have the following bound on the expectation of N_T,

$$\mathbb{E}\left[(N_T - 1)^p\right] \leq C(p)\Delta^{-p}, \tag{10}$$

for any $p \in \left[0, \frac{[p_0/2]}{\beta\vee l}\right]$, where $C(p)$ is a positive constant which does not depend on Δ.

By following the argument in the proof of Lemma 2 in [11], we can obtain the estimate (10) as a consequence of Theorem 2.4.

2.4. Convergence

Theorem 2.7. *Let Assumptions **A1–A4** hold and $p_0 \geq 2l\vee(2+4\alpha+4m)$. Suppose that functions b_Δ, σ_Δ satisfy all conditions of Theorem 2.4, and*

$$|b(x) - b_\Delta(x)| \leq L_4\Delta|b(x)|, \quad |\sigma(x) - \sigma_\Delta(x)| \leq L_4\Delta^{1/2}|\sigma(x)|^2, \tag{11}$$

for some constant $L_4 > 0$.

Then, for any $T > 0$, there exists a positive constant $C_T = C(x_0, L, L_1, L_2, L_3, L_4, \gamma, \eta, T)$ such that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|\hat{X}_t - X_t|\right] \leq \begin{cases} C_T\Delta^\alpha & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ \dfrac{C_T}{\log\frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{12}$$

Moreover, if $L_1 < 0$ and $\gamma < 0$, then there exists a positive constant $C = C(x_0, L, L_1, L_2, L_3, L_4, \gamma, \eta)$, which does not depend on T, such that

$$\sup_{t\geq 0}\mathbb{E}\left[|\hat{X}_t - X_t|\right] \leq \begin{cases} C\Delta^{\alpha} & \text{if } 0<\alpha\leq \dfrac{1}{2}, \\ \dfrac{C}{\log\dfrac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{13}$$

The proof of Theorem 2.7 is deferred to Section 3.5.

Remark 2.8. It is straightforward to verify that under Assumptions **A1–A4**, the functions b_Δ and σ_Δ defined in (6) also satisfy condition (11).

3. Proofs

3.1. Proof of Proposition 2.3

Applying Itô's formula for $e^{-p\gamma t}|X_t|^p$ and the Condition **A1**, for any $p\in[2,p_0]$, we have

$$\begin{aligned} e^{-p\gamma t}|X_t|^p &= |x_0|^p + \int_0^t pe^{-p\gamma s}|X_s|^{p-2}\left[-\gamma|X_s|^2 + X_s b(X_s) + \frac{p-1}{2}\sigma^2(X_s)\right]ds \\ &\quad + \int_0^t pe^{-p\gamma s}|X_s|^{p-1}\sigma(X_s)dW_s \\ &\leq |x_0|^p + \int_0^t p\eta e^{-p\gamma s}|X_s|^{p-2}ds + \int_0^t pe^{-p\gamma s}|X_s|^{p-1}\sigma(X_s)dW_s. \end{aligned} \tag{14}$$

For each $N>0$, we consider $\tau_N = \inf\{t\geq 0 : |X_t|\geq N\}$. It follows from (14) that

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq |x_0|^p + \int_0^t p\eta\mathbb{E}\left[e^{-p\gamma(s\wedge\tau_N)}|X_{s\wedge\tau_N}|^{p-2}\right]ds.$$

Since $(p-2)|x|^p + 2\geq p|x|^{p-2}$, we get

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq |x_0|^p + 2\eta t e^{p|\gamma|t} + (p-2)\eta\int_0^t \mathbb{E}\left[e^{-p\gamma(s\wedge\tau_N)}|X_{s\wedge\tau_N}|^p\right]ds.$$

It then follows from Gronwall's inequality that

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq \left(|x_0|^p + 2\eta t e^{p|\gamma|t}\right)e^{(p-2)\eta t}.$$

This implies

$$\mathbb{P}[\tau_N < t] \leq \left(|x_0|^p + 2\eta t e^{p|\gamma|t}\right)e^{(p-2)\eta t}e^{p|\gamma|t}N^{-p}.$$

Thus, $\tau_N\uparrow\infty$ as $N\to\infty$. It follows from (14) that

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq |x_0|^p + \mathbb{E}\left[\int_0^{t\wedge\tau_N} p\eta e^{-p\gamma s}|X_s|^{p-2}ds\right]. \tag{15}$$

Let $N\to\infty$ in (15), and using Fatou's lemma for the left hand side, and the monotone convergence theorem for the right hand side, we get

$$\mathbb{E}\left[e^{-p\gamma t}|X_t|^p\right] \le |x_0|^p + \int_0^t p\eta e^{-p\gamma s}\mathbb{E}\left[|X_s|^{p-2}\right]ds. \tag{16}$$

Let $p=2$, we get

$$\mathbb{E}\left[e^{-2\gamma t}|X_t|^2\right] \le |x_0|^2 + \int_0^t 2\eta e^{-2\gamma s}ds,$$

which implies (7) for $p=2$. Furthermore, thanks to Hölder's inequality, (7) holds for any $p \in [0,2]$.

Suppose that (7) holds for some $p = p_1 \in [0, p_0 - 2]$, i.e.,

$$\mathbb{E}\left[|X_t|^{p_1}\right] \le \begin{cases} |x_0^2 e^{2\gamma t} + \frac{\eta}{\gamma}(e^{2\gamma t}-1)|^{p_1/2} & \text{if } \gamma \ne 0, \\ |x_0^2 + 2\eta t|^{p_1/2} & \text{if } \gamma = 0. \end{cases} \tag{17}$$

We shall show that (7) holds for $p = p_1 + 2$. Thanks to the estimate (16), we have

$$\mathbb{E}\left[e^{-(p_1+2)\gamma t}|X_t|^{p_1+2}\right] \le |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta e^{-(p_1+2)\gamma s}\mathbb{E}\left[|X_s|^{p_1}\right]ds.$$

<u>If $\gamma = 0$:</u> using the estimate (17) to obtain

$$\begin{aligned}\mathbb{E}\left[|X_t|^{p_1+2}\right] &\le |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta|x_0^2 + 2\eta s|^{p_1/2}ds. \\ &= |x_0|^{p_1+2} + \left(|x_0^2 + 2\eta t|^{(p_1+2)/2} - |x_0^2|^{(p_1+2)/2}\right) \\ &= |x_0^2 + 2\eta t|^{(p_1+2)/2}.\end{aligned}$$

<u>If $\gamma \ne 0$:</u> using the estimate (17) to obtain

$$\begin{aligned}\mathbb{E}\left[e^{-(p_1+2)\gamma t}|X_t|^{p_1+2}\right] &\le |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta e^{-(p_1+2)\gamma s}\left|x_0^2 e^{2\gamma s} + \frac{\eta}{\gamma}(e^{2\gamma s}-1)\right|^{p_1/2}ds. \\ &= |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta e^{-2\gamma s}\left|x_0^2 + \frac{\eta}{\gamma} - \frac{\eta}{\gamma}e^{-2\gamma s}\right|^{p_1/2}ds. \\ &= |x_0|^{p_1+2} + \int_0^t \frac{1}{2}(p_1+2)\left|x_0^2 + \frac{\eta}{\gamma} - \frac{\eta}{\gamma}e^{-2\gamma s}\right|^{p_1/2} d\left(x_0^2 + \frac{\eta}{\gamma} - \frac{\eta}{\gamma}e^{-2\gamma s}\right). \\ &= |x_0|^{p_1+2} + \left|x_0^2 + \frac{\eta}{\gamma} - \frac{\eta}{\gamma}e^{-2\gamma t}\right|^{(p_1+2)/2} - |x_0^2|^{(p_1+2)/2} \\ &= \left|x_0^2 + \frac{\eta}{\gamma} - \frac{\eta}{\gamma}e^{-2\gamma t}\right|^{(p_1+2)/2}.\end{aligned}$$

This implies

$$\mathbb{E}\left[|X_t|^{p_1+2}\right] \le \left|x_0^2 e^{2\gamma t} + \frac{\eta}{\gamma}e^{2\gamma t} - \frac{\eta}{\gamma}\right|^{(p_1+2)/2} = \left|x_0^2 e^{2\gamma t} + \frac{\eta}{\gamma}(e^{2\gamma t}-1)\right|^{(p_1+2)/2}.$$

By the induction principle, we obtain (7) for any $p \in [0, p_0]$.

3.2. Yamada and Watanabe approximation

We recall the approximation technique of Yamada and Watanabe (see [13, 15]). For each $\delta > 1$ and $\varepsilon > 0$ there exists a continuous function $\psi_{\delta\varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with supp$\psi_{\delta\varepsilon} \subset [\varepsilon/\delta; \varepsilon]$ such that

$$\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta\varepsilon}(z)dz = 1; \quad 0 \le \psi_{\delta\varepsilon}(z) \le \frac{2}{z \log \delta}, \quad z > 0.$$

Define

$$\phi_{\delta\varepsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta\varepsilon}(z)dzdy, \quad x \in \mathbb{R}.$$

It is easy to verify that $\phi_{\delta\epsilon}$ has the following useful properties: for any $x \in \mathbb{R}$,

YW1. $\phi'_{\delta\varepsilon}(x) = \frac{x}{|x|}\phi'_{\delta\varepsilon}(|x|)$,
YW2. $0 \le |\phi'_{\delta\varepsilon}(x)| \le 1$,
YW3. $|x| \le \varepsilon + \phi_{\delta\varepsilon}(x)$,
YW4. $\frac{\phi'_{\delta\varepsilon}(|x|)}{|x|} \le \frac{\delta}{\varepsilon}$,
YW5. $\phi''_{\delta\varepsilon}(|x|) = \psi_{\delta\varepsilon}(|x|) \le \frac{2}{|x| \log \delta} 1_{[\frac{\varepsilon}{\delta};\varepsilon]}(|x|) \le \frac{2\delta}{\varepsilon \log \delta}$.

3.3. Proof of Proposition 2.1

Throughout the proof, we use the following result, which is a consequence of the strong Markov property of W,

$$\mathbb{E}\left[(W_s - W_{\underline{s}})^r | \mathcal{F}_{\underline{s}}\right] = \begin{cases} 0 & \text{if } r \text{ is an odd integer} \\ \alpha_r (s - \underline{s})^{r/2} & \text{if } r \text{ is an even integer,} \end{cases} \tag{18}$$

for some positive constant α_r.

Proof of Proposition 2.1. We also use the projection method like in [11]. However, to deal with the superlinear growth of σ, we use the Yamada-Watanabe function $\phi_{\delta\epsilon}$ instead of L^p-norm.

For each $H > |x_0|$, we define a projected approximation scheme as follows:

$$\begin{cases} t_0^H = 0, \quad t_{k+1}^H = t_k^H + h_\Delta(\hat{X}^H_{t_k^H}), \\ \hat{X}^H_{t_{k+1}^H} = P_H\left(\hat{X}^H_{t_k^H} + b_\Delta\left(\hat{X}^H_{t_k^H}\right) h_\Delta(\hat{X}^H_{t_k^H}) + \sigma_\Delta\left(\hat{X}^H_{t_k^H}\right)(W_{t_{k+1}^H} - W_{t_k^H})\right), \end{cases}$$

where $P_H(Y) \triangleq \min(1, H/|Y|)Y$ and therefore $|\hat{X}^H_{t_k^H}| \le H$ for all k. Thus $h_\Delta(\hat{X}^H_{t_k^H}) \ge C(H, L, l, m)\Delta$, which implies that $t_k^H \uparrow \infty$ as $k \to \infty$. We also note that for each k, t_k^H is a stopping time and t_{k+1}^H is $\mathcal{F}_{t_k^H}$-measurable. Set $\underline{t}^H = \max\{t_k^H : t_k^H \le t\}$. Then $\underline{t}^H$ is also a stopping time.

The piecewise constant approximation for intermediate times is again $\bar{X}_t^H = \hat{X}^H_{\underline{t}^H}$, and the continuous approximation is

$$\hat{X}_t^H = P_H\left(\hat{X}_{\underline{t}^H}{}^H + b_\Delta\left(\hat{X}_{\underline{t}^H}{}^H\right)(t-\underline{t}^H) + \sigma_\Delta\left(\hat{X}_{\underline{t}^H}{}^H\right)\left(W_t - W_{\underline{t}^H}\right)\right)$$
$$= P_H\left(\bar{X}_t^H + b_\Delta\left(\bar{X}_t^H\right)(t-\underline{t}^H) + \sigma_\Delta\left(\bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)\right).$$

Firstly, we note that $P_H(Y) \triangleq \min(1, H/|Y|)Y$ implies $\phi_{\delta\varepsilon}(P_H(Y)) \le \phi_{\delta\varepsilon}(Y)$. Hence,

$$\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) = \phi_{\delta\varepsilon}\left(P_H\left(\bar{X}_t^H + b_\Delta\left(\bar{X}_t^H\right)(t-\underline{t}^H) + \sigma_\Delta\left(\bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)\right)\right)$$
$$\le \phi_{\delta\varepsilon}\left(\bar{X}_t^H + b_\Delta\left(\bar{X}_t^H\right)(t-\underline{t}^H) + \sigma_\Delta\left(\bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)\right).$$

Using Taylor's expansion, there exists an $(\mathcal{F}_t)$-adapted process (ξ_t) such that

$$\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \le \phi_{\delta\varepsilon}\left(\bar{X}_t^H\right) + \phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)\left(b_\Delta\left(\bar{X}_t^H\right)(t-\underline{t}^H) + \sigma_\Delta\left(\bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)\right)$$
$$+\frac{1}{2}\phi''_{\delta\varepsilon}(\xi_t)\left(b_\Delta\left(\bar{X}_t^H\right)(t-\underline{t}^H) + \sigma_\Delta\left(\bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)\right)^2.$$

Note that $|\phi''_{\delta\varepsilon}(x)| \le \frac{2\delta}{\varepsilon\log\delta}$, thus

$$\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \le \phi_{\delta\varepsilon}\left(\bar{X}_t^H\right) + \phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)\left[b_\Delta\left(\bar{X}_t^H\right) - b_\Delta(0)\right](t-\underline{t}^H) + \phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)b_\Delta(0)(t-\underline{t}^H)$$
$$+ \phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)\sigma_\Delta\left(\bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)$$
$$+\frac{2\delta}{\varepsilon\log\delta}\left(b_\Delta^2\left(\bar{X}_t^H\right)(t-\underline{t}^H)^2 + \sigma_\Delta^2\left(\bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)^2\right).$$

Thanks to **T3**, we get

$$b_\Delta^2(\bar{X}_t^H)(t-\underline{t}^H) \le L^2 b^2\left(\bar{X}_t^H\right)\frac{\Delta}{b^2\left(\bar{X}_t^H\right)} = L^2\Delta.$$

It follows from **YW3** and **T2** that

$$\phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)\left[b_\Delta\left(\bar{X}_t^H\right) - b_\Delta(0)\right](t-\underline{t}^H) = \frac{\phi'_{\delta\varepsilon}\left(|\bar{X}_t^H|\right)}{|\bar{X}_t^H|}\bar{X}_t^H\left[b_\Delta\left(\bar{X}_t^H\right) - b_\Delta(0)\right](t-\underline{t}^H)$$
$$\le L|\bar{X}_t^H|\phi'_{\delta\varepsilon}\left(|\bar{X}_t^H|\right)(t-\underline{t}^H).$$

Because $|x|\phi'_{\delta\varepsilon}(|x|) \le \phi_{\delta\varepsilon}(x) + \varepsilon$,

$$\phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)\left[b_\Delta\left(\bar{X}_t^H\right) - b_\Delta(0)\right](t-\underline{t}^H) \le L\left(\phi_{\delta\varepsilon}\left(\bar{X}_t^H\right) + \varepsilon\right)(t-\underline{t}^H).$$

Moreover, thanks to **T4** and the fact that $dW_t^2 = 2W_t dW_t + dt$, we have

$$\sigma_\Delta^2\left(\underline{t}^H; \bar{X}_t^H\right)\left(W_t - W_{\underline{t}^H}\right)^2 \le \frac{L^2}{\Delta}\left(2\int_{\underline{t}^H}^t \left(W_s - W_{\underline{s}^H}\right)dW_s + (t-\underline{t}^H)\right).$$

Hence,

$$\begin{aligned}\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \le& \left[1+L\left(t-\underline{t}^H\right)\right]\phi_{\delta\varepsilon}\left(\bar{X}_t^H\right)\\ &+\left(L\varepsilon+|b_\Delta(0)|+\frac{2L^2\delta\Delta}{\varepsilon\log\delta}+\frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)(t-\underline{t}^H)+\int_{\underline{t}^H}^t\phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right)\sigma_\Delta\left(\bar{X}_s^H\right)dW_s\\ &+\int_{\underline{t}^H}^t\frac{4L^2\delta}{\varepsilon\Delta\log\delta}\left(W_s-W_{\underline{s}^H}\right)dW_s.\end{aligned} \tag{19}$$

Note that $e^{-Lt}(1+L(t-\underline{t}^H))\le e^{-L\underline{t}^H}$. By multiplying e^{-Lt} to both sides of (19), we have

$$\begin{aligned}e^{-Lt}\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right)\le& M_t+e^{-L\underline{t}^H}\phi_{\delta\varepsilon}\left(\bar{X}_t^H\right)+\left(L\varepsilon+|b_\Delta(0)|+\frac{2L^2\delta\Delta}{\varepsilon\log\delta}+\frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)e^{-Lt}(t-\underline{t}^H)\\ &+\left[e^{-Lt}-e^{-L\left(\underline{t}^H+h\left(\bar{X}_t^H\right)\right)}\right]\left(\phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)\sigma_\Delta\left(\bar{X}_t^H\right)(W_t-W_{\underline{t}^H})+\frac{4L^2\delta}{\varepsilon\log\delta\Delta}\left[(W_t-W_{\underline{t}^H})^2-\left(t-\underline{t}^H\right)\right]\right),\end{aligned} \tag{20}$$

where

$$M_t=\int_{\underline{t}^H}^t e^{-L\left(\underline{s}^H+h\left(\bar{X}_s^H\right)\right)}\left(\phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right)\sigma_\Delta\left(\bar{X}_s^H\right)+\frac{4L^2\delta}{\varepsilon\Delta\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right)dW_s.$$

Let $\omega(t,\Delta)$ be the modulus of continuity of W, i.e.,

$$\omega(t,\Delta)=\sup_{s_1,s_2\in[0,t];|s_2-s_1|\le\Delta}|W(s_1)-W(s_2)|.$$

For any $p>0$, it follows from Theorem 1 in [16] that

$$\mathbb{E}\left[|\omega(t,\Delta)|^p\right]\le C_p(\Delta\log\frac{2t}{\Delta})^{p/2}. \tag{21}$$

Therefore,

$$\begin{aligned}e^{-Lt}\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right)\le& M_t+e^{-L\underline{t}^H}\phi_{\delta\varepsilon}\left(\bar{X}_t^H\right)+\left(L\varepsilon+|b_\Delta(0)|+\frac{2L^2\delta\Delta}{\varepsilon\log\delta}+\frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)\int_{\underline{t}^H}^t e^{-Ls}ds\\ &+Le^{-L(t-\Delta)}(t-\underline{t}^H)\left(\frac{L}{\sqrt{\Delta}}\omega(t,\Delta)+\frac{4L^2\delta}{\varepsilon\Delta\log\delta}\omega(t,\Delta)^2\right).\end{aligned} \tag{22}$$

It also follows from (20) that

$$\begin{aligned}e^{-Lt_{k+1}^H}\phi_{\delta\varepsilon}\left(\hat{X}_{t_k^H}\right)\le& M_{t_k^H}+e^{-Lt_k^H}\phi_{\delta\varepsilon}\left(\hat{X}_{t_k^H}^H\right)\\ &+\left(L\varepsilon+|b_\Delta(0)|+\frac{2L^2\delta\Delta}{\varepsilon\log\delta}+\frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)\int_{t_k^H}^{t_{k+1}^H}e^{-Ls}ds,\end{aligned} \tag{23}$$

where

$$M_{t_k^H} = \int_{t_k^H}^{t_{k+1}^H} e^{-L\left(\underline{s}^H + h\left(\bar{X}_s^H\right)\right)} \left(\phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right) \sigma_\Delta\left(\bar{X}_s^H\right) + \frac{4L^2\delta}{\varepsilon \log \delta\Delta} \left(W_s - W_{\underline{s}^H}\right) \right) dW_s.$$

Summing (23) over multiple timesteps and then adding (22) gives

$$\begin{aligned} e^{-Lt}|\hat{X}_t^H| &\le \varepsilon + e^{-Lt}\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \\ &\le \varepsilon + \bar{M}_t + \phi_{\delta\varepsilon}(x_0) + \left(L\varepsilon + |b_\Delta(0)| + \frac{2L^2\delta\Delta}{\varepsilon\log\delta} + \frac{2L^2\delta}{\varepsilon\Delta\log\delta} \right) \int_0^t e^{-Ls} ds \\ &\quad + Le^{-L(t-\Delta)}(t - \underline{t}^H) \left(\frac{L}{\sqrt{\Delta}}\omega(t,\Delta) + \frac{4L^2\delta}{\varepsilon\Delta\log\delta}\omega(t,\Delta)^2 \right), \end{aligned}$$

where

$$\bar{M}_t = \int_0^t e^{-L\left(\underline{s}^H + h\left(\bar{X}_s^H\right)\right)} \left(\phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right) \sigma_\Delta\left(\bar{X}_s^H\right) + \frac{4L^2\delta}{\varepsilon \log \delta\Delta} \left(W_s - W_{\underline{s}^H}\right) \right) dW_s.$$

Hence, for any the stopping time $\tau \le t$, it follows from (21) that there exists a constant $C = C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))$, which does not depend on H such that

$$\mathbb{E}\left[e^{-L\tau}|\hat{X}_\tau^H|\right] \le C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))\left(1 + |\log t|\right).$$

Applying Proposition IV.4.7 in [17], for any $p \in (0,1)$, we have

$$\mathbb{E}\left[\sup_{0\le s\le t} e^{-Lps}|\hat{X}_\tau^H|^p\right] \le \frac{2-p}{1-p} C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))^p \left(1 + |\log t|\right)^p.$$

Hence,

$$\mathbb{E}\left[\sup_{0\le s\le t} |\hat{X}_\tau^H|^p\right] \le \frac{2-p}{1-p} C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))^p \left(1 + |\log t|\right)^p e^{Lpt}. \tag{24}$$

On the other hand, for any $T > 0$,

$$\mathbb{P}[t_k < T] = \mathbb{P}\left[t_k < T, \sup_{0\le s\le T} |\hat{X}_s^H| > \frac{H}{2}\right] + \mathbb{P}\left[t_k < T, \sup_{0\le s\le T} |\hat{X}_s^H| \le \frac{H}{2}\right].$$

It follows from Markov's inequality and (24) that

$$\begin{aligned} \mathbb{P}\left[t_k < T, \sup_{0\le s\le T} |\hat{X}_s^H| > \frac{H}{2}\right] &\le \mathbb{P}\left[\sup_{0\le s\le T} |\hat{X}_s^H| > \frac{H}{2}\right] \le \left(\frac{2}{H}\right)^p \mathbb{E}\left[\sup_{0\le s\le t} |\hat{X}_\tau^H|^p\right] \\ &\le \left(\frac{2}{H}\right)^p \frac{2-p}{1-p} C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))^p \left(1 + |\log T|\right)^p e^{LpT}. \end{aligned}$$

On the other hand, on the set $\left\{\sup_{0\le s\le T} |\hat{X}_s^H| \le \frac{H}{2}\right\}, \hat{X}_s^H = \hat{X}_s$ for all $s \le T$, which, in turn, implies that $t_k^H = t_k$ if $t_k < T$. Hence,

$$\limsup_{k\to\infty} \mathbb{P}\left[t_k < T,\ \sup_{0\le s\le T} |\hat{X}^H_s| \le \frac{H}{2}\right] \le \limsup_{k\to\infty} \mathbb{P}\left[t^H_k < T\right] = 0,$$

which implies that

$$\limsup_{k\to\infty} \mathbb{P}[t_k < T] \le \left(\frac{2}{H}\right)^p \frac{2-p}{1-p} C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))^p \left(1 + |\log T|\right)^p e^{LpT},$$

for any $H > 0$. Let $H \to \infty$, we get $\limsup_{k\to\infty} \mathbb{P}[t_k < T] = 0$ for any $T > 0$. This implies that $\lim_{k\to\infty} t_k = +\infty$ almost surely, which is the desired result.

3.4. Proof of Theorem 2.4

We start will the following key estimate on moments of $\hat{X}$.

Lemma 3.1. *Assume that Conditions **T1–T4** hold. Then for any $p > 0$ and $T > 0$, there exists a positive constant $C(p, L, T, x_0, \Delta) < \infty$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |\hat{X}_t|^p\right] \le C(p, L, T, x_0, \Delta).$$

Proof. Applying Itô's formula for $e^{-Lt}\phi_{\delta\varepsilon}(\hat{X}_t)$ gives

$$\begin{aligned} e^{-Lt}|\hat{X}_t| &\le e^{-Lt}\varepsilon + e^{-Lt}\phi_{\delta\varepsilon}(\hat{X}_t) \\ &\le \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\left[-L\phi_{\delta\varepsilon}(\hat{X}_s) + \phi'_{\delta\varepsilon}(\hat{X}_s)b_\Delta(\bar{X}_s) + \frac{1}{2}\phi''_{\delta\varepsilon}(\hat{X}_s)|\sigma_\Delta(\bar{X}_s)|^2\right] ds \\ &\quad + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\hat{X}_s)\sigma_\Delta(\bar{X}_s)dW_s. \end{aligned}$$

Applying Taylor's expansion for $\phi'_{\delta\varepsilon}$, there exists an $(\mathcal{F}_s)$-adapted process $\xi = (\xi_s)$ such that

$$\begin{aligned} \phi'_{\delta\varepsilon}(\hat{X}_s)b_\Delta(\bar{X}_s) &= \left(\phi'_{\delta\varepsilon}(\bar{X}_s) + \phi''_{\delta\varepsilon}(\xi_s)\left(\hat{X}_s - \bar{X}_s\right)\right)b_\Delta(\bar{X}_s) \\ &= \phi'_{\delta\varepsilon}(\bar{X}_s)b_\Delta(\bar{X}_s) + \phi''_{\delta\varepsilon}(\xi_s)\left(b_\Delta(\bar{X}_s)(s - \underline{s}) + \sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\right)b_\Delta(\bar{X}_s) \\ &= \phi'_{\delta\varepsilon}(\bar{X}_s)\left(b_\Delta(\bar{X}_s) - b_\Delta(0)\right) + \phi'_{\delta\varepsilon}(\bar{X}_s)b_\Delta(0) \\ &\quad + \phi''_{\delta\varepsilon}(\xi_s)\left(b^2_\Delta(\bar{X}_s)(s - \underline{s}) + b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\right) \\ &= \frac{\phi'_{\delta\varepsilon}(|\bar{X}_s|)}{|\bar{X}_s|}\bar{X}_s\left(b_\Delta(\bar{X}_s) - b_\Delta(0)\right) + \phi'_{\delta\varepsilon}(\bar{X}_s)b_\Delta(0) \\ &\quad + \phi''_{\delta\varepsilon}(\xi_s)\left(b^2_\Delta(\bar{X}_s)(s - \underline{s}) + b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\right) \\ &\le L|\bar{X}_s| + |b_\Delta(0)| + \frac{2\delta}{\varepsilon\log\delta}C\Delta + \frac{2\delta}{\varepsilon\log\delta}|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|. \end{aligned}$$

Hence,

$$e^{-Lt}|\hat{X}_t| \leq \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\Big[-L\phi_{\delta\varepsilon}(\hat{X}_s) + L|\bar{X}_s| + |b_\Delta(0)| + \frac{2\delta}{\varepsilon\log\delta}C\Delta + \frac{2\delta}{\varepsilon\log\delta}|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})| + \frac{\delta}{\varepsilon\log\delta}\frac{C}{\Delta}\Big]ds + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\hat{X}_s)\sigma_\Delta(\bar{X}_s)dW_s.$$

Note that

$$\begin{aligned}-\phi_{\delta\varepsilon}(\hat{X}_s) + |\bar{X}_s| &\leq -|\hat{X}_s| + \varepsilon + |\bar{X}_s| \\ &\leq \varepsilon + |b_\Delta(\bar{X}_s)(s - \underline{s})| + |\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|.\end{aligned}$$

Thus,

$$\begin{aligned}e^{-Lt}|\hat{X}_t| \leq{}& (1+Lt)\varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t Le^{-Ls}\Big(|b_\Delta(\bar{X}_s)(s-\underline{s})| + |\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|\Big)ds \\ &+ \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\hat{X}_s)\sigma_\Delta(\bar{X}_s)dW_s \\ &+ \int_0^t e^{-Ls}\Big[|b_\Delta(0)| + \frac{2\delta}{\varepsilon\log\delta}C\Delta + \frac{2\delta}{\varepsilon\log\delta}|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})| + \frac{\delta}{\varepsilon\log\delta}\frac{C}{\Delta}\Big]ds.\end{aligned} \tag{25}$$

Note that, for any $p > 0$, there exists a constant $C(p) > 0$ such that

$$\begin{aligned}&\mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|^p\Big] \\ &= \mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)|^p\mathbb{E}\Big[|W_s - W_{\underline{s}}|^p|\mathcal{F}_{\underline{s}}\Big]\Big] \\ &\leq C(p)\mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)|^p(s-\underline{s})^{p/2}\Big].\end{aligned}$$

Thanks to (3) and Condition **T4**, we have

$$\mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|^p\Big] \leq C(p, L).$$

Therefore, by choosing $\varepsilon = 1, \delta = 2$ in (25), it follows from **T1–T4**, Hölder's inequality and Burkholder-Davis-Gundy's inequality that for any $T > 0$, there exists a positive constant $C(p, L, T, x_0, \Delta) < \infty$ such that

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|\hat{X}_t|^p\Big] \leq C(p, L, T, x_0, \Delta).$$

□

Proof of Theorem 2.4. Thanks to Hölder's inequality, it is sufficient to show (8) for k is a positive interger and $k \leq p_0/2$. We will use the induction method. Firstly, for $k = 1$, applying Itô's formula for $e^{-2\gamma t}\hat{X}_t^2$, we have

$$e^{-2\gamma t}\hat{X}_t^2 = x_0^2 + \int_0^t e^{-2\gamma s}\Big(-2\gamma\hat{X}_s^2 + 2\hat{X}_s b_\Delta(\bar{X}_s) + \sigma_\Delta^2(\bar{X}_s)\Big)ds + \int_0^t 2e^{-2\gamma s}\hat{X}_s\sigma_\Delta(\bar{X}_s)dW_s. \tag{26}$$

On the other hand, it follows from (5) that

$$\hat{X}_s^2 = \bar{X}_s^2 + 2\bar{X}_s b_\Delta(\bar{X}_s)(s-\underline{s}) + b_\Delta^2(\bar{X}_s)(s-\underline{s})^2 + \sigma_\Delta^2(\bar{X}_s)(W_s - W_{\underline{s}})^2 + 2(\bar{X}_s + b_\Delta(\bar{X}_s)(s-\underline{s}))\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}}).$$

Thanks to **T3**, (18), and (3),

$$\max\left\{|\bar{X}_s b_\Delta(\bar{X}_s)(s-\underline{s})|, b_\Delta^2(\bar{X}_s)(s-\underline{s})^2, \mathbb{E}\left[\sigma_\Delta^2(\bar{X}_s)(W_s - W_{\underline{s}})^2|\mathcal{F}_{\underline{s}}\right]\right\} \le C\Delta. \tag{27}$$

Therefore,

$$\mathbb{E}\left[-2\gamma\hat{X}_s^2\right] \le \mathbb{E}\left[-2\gamma\bar{X}_s^2\right] + C|\gamma|\Delta. \tag{28}$$

A similar argument yields

$$\mathbb{E}\left[2\hat{X}_s b_\Delta(\bar{X}_s)\right] \le \mathbb{E}\left[2\bar{X}_s b_\Delta(\bar{X}_s)\right] + C|\gamma|\Delta. \tag{29}$$

It then follows from (8), (26), (28), (29), that

$$\mathbb{E}\left[e^{-2\gamma t}\hat{X}_t^2\right] \le |x_0|^2 + C(\eta + |\gamma|\Delta)\int_0^t e^{-2\gamma s}ds. \tag{30}$$

Note that $\bar{X}_t = \hat{X}_t - b_\Delta(\bar{X}_t)(t-\underline{t}) - \sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})$, which together with (27) implies the following estimate for any $p > 0$,

$$\begin{aligned}\mathbb{E}\left[|\bar{X}_t|^p\right] &\le 3^{p-1}\left(\mathbb{E}\left[|\hat{X}_t|^p\right] + \mathbb{E}\left[|b_\Delta(\bar{X}_t)(t-\underline{t})|^p\right] + \mathbb{E}\left[|\sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})|^p\right]\right)\\ &\le 3^{p-1}\left(\mathbb{E}\left[|\hat{X}_t|^p\right] + C\Delta^p + C\Delta^{p/2}\right).\end{aligned} \tag{31}$$

It follows from (30) and (31) that (9) holds for $k = 1$.

Secondly, we assume that (9) holds for any $k \le k_0 \le [p_0/2] - 1$, we will show that (9) still holds for $k = k_0 + 1$.

By applying Itô's formula for $e^{-p\gamma s}\hat{X}_t^p$ with $p = 2(k_0+1)$ being an even integer, we have

$$\begin{aligned}e^{-p\gamma t}|\hat{X}_t|^p = |x_0|^p &+ \int_0^t pe^{-p\gamma s}\left(-\gamma|\hat{X}_s|^p + \hat{X}_s^{p-1}b_\Delta(\bar{X}_s) + \frac{p-1}{2}|\hat{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right)ds\\ &+ \int_0^t pe^{-p\gamma s}|\hat{X}_s|^{p-2}\hat{X}_s\sigma_\Delta(\bar{X}_s)dW_s.\end{aligned} \tag{32}$$

It follows from (5) and the Newton expansion formula that for any positive integer q,

$$|\hat{X}_s|^q = \sum_{0\le i,j,r\le q,\, i+j+r=q} \frac{q!}{i!j!r!}(\bar{X}_s)^i\left(b_\Delta(\bar{X}_s)(s-\underline{s})\right)^j\left(\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\right)^r. \tag{33}$$

Thanks to (18), we have

$$\begin{aligned}\mathbb{E}\left[-\gamma|\hat{X}_s|^p|\mathcal{F}_{\underline{s}}\right] &= -\gamma|\bar{X}_s|^p - p\gamma\bar{X}_s b_\Delta(\bar{X}_s)(s-\underline{s})|\bar{X}_s|^{p-2}\\ &\quad - \sum_{0\le i\le p-2,\, i+j+2r=p}\frac{\gamma q!}{i!j!k!}(\bar{X}_s)^i\left(b_\Delta(\bar{X}_s)(s-\underline{s})\right)^j\left(\sigma_\Delta^2(\bar{X}_s)(s-\underline{s})\right)^r\alpha_{2r}.\end{aligned}$$

Thanks to (27), we get

$$\mathbb{E}\big[-\gamma|\hat{X}_s|^p\big] \le \mathbb{E}\big[-\gamma|\bar{X}_s|^p\big] + C|\gamma|\sum_{i=0}^{p-2}\mathbb{E}\big[|\bar{X}_s|^i\big]. \tag{34}$$

Choose $q = p-1$ and $q = p-2$ in (33), by the same argument, we also have

$$\mathbb{E}\Big[\hat{X}_s^{p-1}b_\Delta(\bar{X}_s)\Big] \le \bar{X}_s^{p-1}b_\Delta(\bar{X}_s) + C\sum_{i=0}^{p-2}\mathbb{E}\big[|\bar{X}_s|^i\big] \tag{35}$$

and

$$\mathbb{E}\left[\frac{p-1}{2}|\hat{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right] \le \frac{p-1}{2}|\bar{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s) + C\sum_{i=0}^{p-2}\mathbb{E}\big[|\bar{X}_s|^i\big]. \tag{36}$$

Combining (34)–(36) to get

$$\begin{aligned}
&\mathbb{E}\left[-\gamma|\hat{X}_s|^p + \hat{X}_s^{p-1}b_\Delta(\bar{X}_s) + \frac{p-1}{2}|\hat{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right]\\
&\quad\le \mathbb{E}\left[-\gamma|\bar{X}_s|^p + \bar{X}_s^{p-1}b_\Delta(\bar{X}_s) + \frac{p-1}{2}|\bar{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right] + C\sum_{i=0}^{p-2}\mathbb{E}\big[|\bar{X}_s|^i\big]\\
&\quad\le C\sum_{i=0}^{p-2}\mathbb{E}\big[|\bar{X}_s|^i\big].
\end{aligned} \tag{37}$$

From (32), (37), (31) and the inductive assumption, we obtain that (9) holds for $k = k_0 + 1,$ which implies the desired result.

3.5. Proof of Theorem 2.7

We need the following uniformly in time bound for the difference between $\hat{X}$ and $\bar{X}$.

Lemma 3.2. *Suppose that coefficients b_Δ, σ_Δ satisfy all conditions of Theorem 2.4, then there exists a positive constant $C_p = C(p, L)$ such that*

$$\sup_{t\ge 0}\mathbb{E}\big[|\hat{X}_t - \bar{X}_t|^p\big] \le C_p\Delta^{p/2}, \tag{38}$$

for any $p \ge 0$.

Proof. From (5),

$$\begin{aligned}
|\hat{X}_t - \bar{X}_t|^p &= |b_\Delta(\bar{X}_t)(t-\underline{t}) + \sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})|^p\\
&\le 2^{p-1}\Big(|b_\Delta(\bar{X}_t)(t-\underline{t})|^p + |\sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})|^p\Big)\\
&\le 2^{p-1}\Big(|b_\Delta(\bar{X}_t)|^p|h_\Delta(\bar{X}_t)|^p + |\sigma_\Delta(\bar{X}_t)|^p|W_t - W_{\underline{t}}|^p\Big).
\end{aligned}$$

By applying **T3** and (3), we have

$$|b_\Delta(\bar{X}_t)h_\Delta(\bar{X}_t)| \le \frac{L\Delta}{4} \quad \text{and} \quad |\sigma_\Delta(\bar{X}_t)|h_\Delta(\bar{X}_t)|^{1/2}| \le L\Delta^{1/2}.$$

Using (18), we obtain the desired result. □

Proof of Theorem 2.7. Put $Y_t = X_t - \hat{X}_t$. Applying the property **YW3** and Itô's formula for $e^{-L_1 t}\phi_{\delta\varepsilon}(Y_t)$ gives

$$\begin{aligned} e^{-L_1 t}|Y_t| &\le e^{-L_1 t}\varepsilon + e^{-L_1 t}\phi_{\delta\varepsilon}(Y_t) \\ &= e^{-L_1 t}\varepsilon + \int_0^t e^{-L_1 s}\Big[-L_1\phi_{\delta\varepsilon}(Y_s) + \phi'_{\delta\varepsilon}(Y_s)\big(b(X_s) - b_\Delta(\bar{X}_s)\big) + \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)|\sigma(X_s) - \sigma_\Delta(\bar{X}_s)|^2\Big]ds \\ &+ \int_0^t e^{-L_1 s}\phi'_{\delta\varepsilon}(Y_s)\big(\sigma(X_s) - \sigma_\Delta(\bar{X}_s)\big)dW_s. \end{aligned} \tag{39}$$

Set $J_1(s) = \phi'_{\delta\varepsilon}(Y_s)(b(X_s) - b_\Delta(\bar{X}_s))$ and $J_2(s) = \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)|\sigma(X_s) - \sigma_\Delta(\bar{X}_s)|^2$. Firstly, we write

$$J_1(s) = \phi'_{\delta\varepsilon}(Y_s)\big(b(X_s) - b(\hat{X}_s)\big) + \phi'_{\delta\varepsilon}(Y_s)\big(b(\hat{X}_s) - b(\bar{X}_s)\big) + \phi'_{\delta\varepsilon}(Y_s)\big(b(\bar{X}_s) - b_\Delta(\bar{X}_s)\big).$$

Thanks to properties **YW1**, **YW2**, assumptions **A2**, **A3** and (11), we have

$$\begin{aligned} J_1(s) &\le \frac{\phi'_{\delta\varepsilon}(|Y_s|)}{|Y_s|}Y_s\big(b(X_s) - b(\hat{X}_s)\big) + |\phi'_{\delta\varepsilon}(Y_s)\big(b(\hat{X}_s) - b(\bar{X}_s)\big)| + |\phi'_{\delta\varepsilon}(Y_s)\big(b(\bar{X}_s) - b_\Delta(\bar{X}_s)\big)| \\ &\le L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + L_2\Big(1 + |\hat{X}_s|^l + |\bar{X}_s|^l\Big)|\hat{X}_s - \bar{X}_s| + C\Delta\Big(1 + |\bar{X}_s|^{l+1}\Big) \\ &\le L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + \frac{3}{2}L_2\Delta^{1/2}\Big(1 + |\hat{X}_s|^{2l} + |\bar{X}_s|^{2l}\Big) + \frac{1}{2}L_2\Delta^{-1/2}|\hat{X}_s - \bar{X}_s|^2 + C\Delta\Big(1 + |\bar{X}_s|^{l+1}\Big). \end{aligned} \tag{40}$$

Secondly, we write

$$J_2(s) = \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)|\sigma(X_s) - \sigma(\hat{X}_s) + \sigma(\hat{X}_s) - \sigma(\bar{X}_s) + \sigma(\bar{X}_s) - \sigma_\Delta(\bar{X}_s)|^2.$$

By using the property **YW5**, the assumption **A4** and (11), we have

$$\begin{aligned}
J_2(s) &\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\big(|\sigma(X_s)-\sigma(\hat{X}_s)|^2+|\sigma(\hat{X}_s)-\sigma(\bar{X}_s)|^2+|\sigma(\bar{X}_s)-\sigma_\Delta(\bar{X}_s)|^2\big)\\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)(L_3^2\big(1+|X_s|^m+|\hat{X}_s|^m\big)^2|X_s-\hat{X}_s|^{1+2\alpha}+\\
&\quad + L_3^2\big(1+|\hat{X}_s|^m+|\bar{X}_s|^m\big)^2|\hat{X}_s-\bar{X}_s|^{1+2\alpha}+L_4\Delta|\sigma(\bar{X}_s)|^4)\\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)(3L_3^2\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)|X_s-\hat{X}_s|^{1+2\alpha}+\\
&\quad + 3L_3^2\big(1+|\hat{X}_s|^{2m}+|\bar{X}_s|^{2m}\big)|\hat{X}_s-\bar{X}_s|^{1+2\alpha}+L_4\Delta|\sigma(\bar{X}_s)|^4)\\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)+\frac{9L_3^2\delta}{\varepsilon\log\delta}\big(1+|\hat{X}_s|^{2m}+|\bar{X}_s|^{2m}\big)|\hat{X}_s-\bar{X}_s|^{2\alpha+1}\\
&\quad +\frac{3L_4\delta\Delta|\sigma(\bar{X}_s)|^4}{\varepsilon\log\delta}\\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\big(1+|\hat{X}_s|^{2m}+|\bar{X}_s|^{2m}\big)^2\Delta^{1/2+\alpha}\\
&\quad +\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-1/2-\alpha}|\hat{X}_s-\bar{X}_s|^{4\alpha+2}+\frac{C\delta\Delta\big(|\bar{X}_s|^{2+4\alpha+4m}+1\big)}{\varepsilon\log\delta}\\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)+\frac{27L_3^2\delta}{2\varepsilon\log\delta}\big(1+|\hat{X}_s|^{4m}+|\bar{X}_s|^{4m}\big)\Delta^{1/2+\alpha}\\
&+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-1/2-\alpha}|\hat{X}_s-\bar{X}_s|^{2+4\alpha}+\frac{C\delta\Delta\big(|\bar{X}_s|^{2+4\alpha+4m}+1\big)}{\varepsilon\log\delta}.
\end{aligned} \tag{41}$$

A combination of (39), (40), (41) and the property $-L_1\phi_{\delta\varepsilon}(x)+L_1\phi'_{\delta\varepsilon}(|x|)|x|\le \max\{L_1\varepsilon;0\}$ implies

$$\begin{aligned}
&\mathbb{E}\big[e^{-L_1t}|Y_t|\big]\\
&\le e^{-L_1t}\varepsilon+\int_0^t e^{-L_1s}[\max\{L_1\varepsilon;0\}+\frac{3}{2}L_2\Delta^{1/2}\Big(1+\mathbb{E}\Big[|\hat{X}_s|^{2l}\Big]+\mathbb{E}\Big[|\bar{X}_s|^{2l}\Big]\Big)\\
&+\frac{1}{2}L_2\Delta^{-1/2}\mathbb{E}\big[|\hat{X}_s-\bar{X}_s|^2\big]+C\Delta\Big(1+\mathbb{E}\Big[|\bar{X}_s|^{l+1}\Big]\Big)+\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+\mathbb{E}\big[|X_s|^{2m}\big]+\mathbb{E}\big[|\hat{X}_s|^{2m}\big]\big)\\
&+\frac{27L_3^2\delta}{2\varepsilon\log\delta}\big(1+\mathbb{E}\big[|\hat{X}_s|^{4m}\big]+\mathbb{E}\big[|\bar{X}_s|^{4m}\big]\big)\Delta^{1/2+\alpha}+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-1/2-\alpha}\mathbb{E}\Big[|\hat{X}_s-\bar{X}_s|^{2+4\alpha}\Big]\\
&+\frac{C\delta\Delta\big(\mathbb{E}\big[|\bar{X}_s|^{2+4\alpha+4m}\big]+1\big)}{\varepsilon\log\delta}\Bigg]ds.
\end{aligned} \tag{42}$$

Thanks to the condition $p_0\ge 2l\vee(2+4\alpha+4m)$, Theorem 2.4, Proposition 2.3, and Lemma 3.2, for any $T>0$, there exists a positive constant C_T such that for any $t\in[0,T]$, it holds that

Table 1. Four SDEs with their parameters.

Case	b	σ	p_0	L_1	γ	η	l	m	α
1	$-1+x-x^3$	$1+(1+x)x^{2/3}$	15	1	-1	18073	2	$\frac{4}{3}$	$\frac{1}{6}$
2	$-1+x-x^3$	$1+\sqrt{\frac{x^4+x^{4/3}}{14}}$	15	1	$\frac{13}{3}$	$\frac{5}{6}$	2	2	$\frac{1}{6}$
3	$-1-x-x^{7/3}$	$1+\sqrt{\frac{2x^2+x^{10/3}+x^{4/3}}{14}}$	15	-1	$\frac{13}{3}$	$\frac{5}{6}$	$\frac{4}{3}$	1	$\frac{1}{6}$
4	$-1-x-x^{7/3}$	$1+\sqrt{\frac{x^{10/3}+x^{4/3}}{14}}$	15	-1	$-\frac{1}{6}$	$\frac{11}{6}$	$\frac{4}{3}$	1	$\frac{1}{6}$

$$\mathbb{E}\left[e^{-L_1 t}|Y_t|\right] \le e^{-L_1 t}\varepsilon + C_T\left[\varepsilon + \Delta^{1/2} + \Delta + \frac{\varepsilon^{2\alpha}}{\log\delta} + \frac{\delta\Delta^{1/2+\alpha}}{\varepsilon\log\delta} + \frac{\delta\Delta}{\varepsilon\log\delta}\right]\int_0^t e^{-L_1 s}ds. \quad (43)$$

If $\alpha \in (0;\frac{1}{2}]$, choosing $\varepsilon = \Delta^{1/2}$, $\delta = 2$, we obtain

$$\sup_{0\le t\le T}\mathbb{E}[|Y_t|] \le C_T\Delta^{\alpha}.$$

If $\alpha = 0$, choosing $\varepsilon = \Delta^{1/4}, \delta = \Delta^{-1/4}$, we obtain

$$\sup_{0\le t\le T}\mathbb{E}[|Y_t|] \le \frac{C_T}{\log\frac{1}{\Delta}}.$$

We obtain (12). Note that if $L_1 < 0$ and $\gamma < 0$, we can choose the constant C_T in (43) such that it does not depend on T. Therefore, we also obtain (13). □

4. Examples

We consider four different SDEs with coefficients given in Table 1. These SDEs are chosen such that: $L_1 > 0, \gamma < 0$ in Case 1; $L_1 > 0, \gamma < 0$ in Case 2; $L_1 < 0, \gamma > 0$ in Case 3; and $L_1 < 0, \gamma < 0$ in Case 4. We choose $X_0 = 0$ in all the cases. It is straightforward to verify that these equations satisfy Assumptions **A1**–**A4** with constants $p_0, L_1, \gamma, \eta, l, m, \alpha$ given in Table 1 as well. In all these cases, $p_0 \ge 2l \vee (2+4\alpha+4m)$, hence it follows from Theorem 2.7 that the tamed-adaptive Euler-Maruyama approximation scheme (2) converges in L^1-norm at the rate of order α in any finite time interval. Moreover, in Case 4, since $L_1 < 0$ and $\gamma < 0$, the tamed-adaptive Euler-Maruyama approximation scheme (2) converges in L^1-norm at the rate of order α in infinite time intervals.

In order to study the empirical rates of convergence of the tamed-adaptive Euler-Maruyama scheme, we consider

$$me(l) = \frac{1}{M}\sum_{k=1}^{M}|\hat{X}_1^{(l,k)} - \hat{X}_1^{(l+1,k)}|,$$

where for each $l \ge 2, (\hat{X}^{(l,k)})_{1\le k\le M}$ is a sequence of independent copies of $\hat{X}^{(l)}$ defined by Equations (2) and (3) with $\Delta = 2^{-l}$. Note that for each k and l, $\hat{X}_1^{(l,k)}$ and $\hat{X}_1^{(l+1,k)}$ must be generated on the same Brownian motion. This can be done by using the Algorithm 1 in [11].

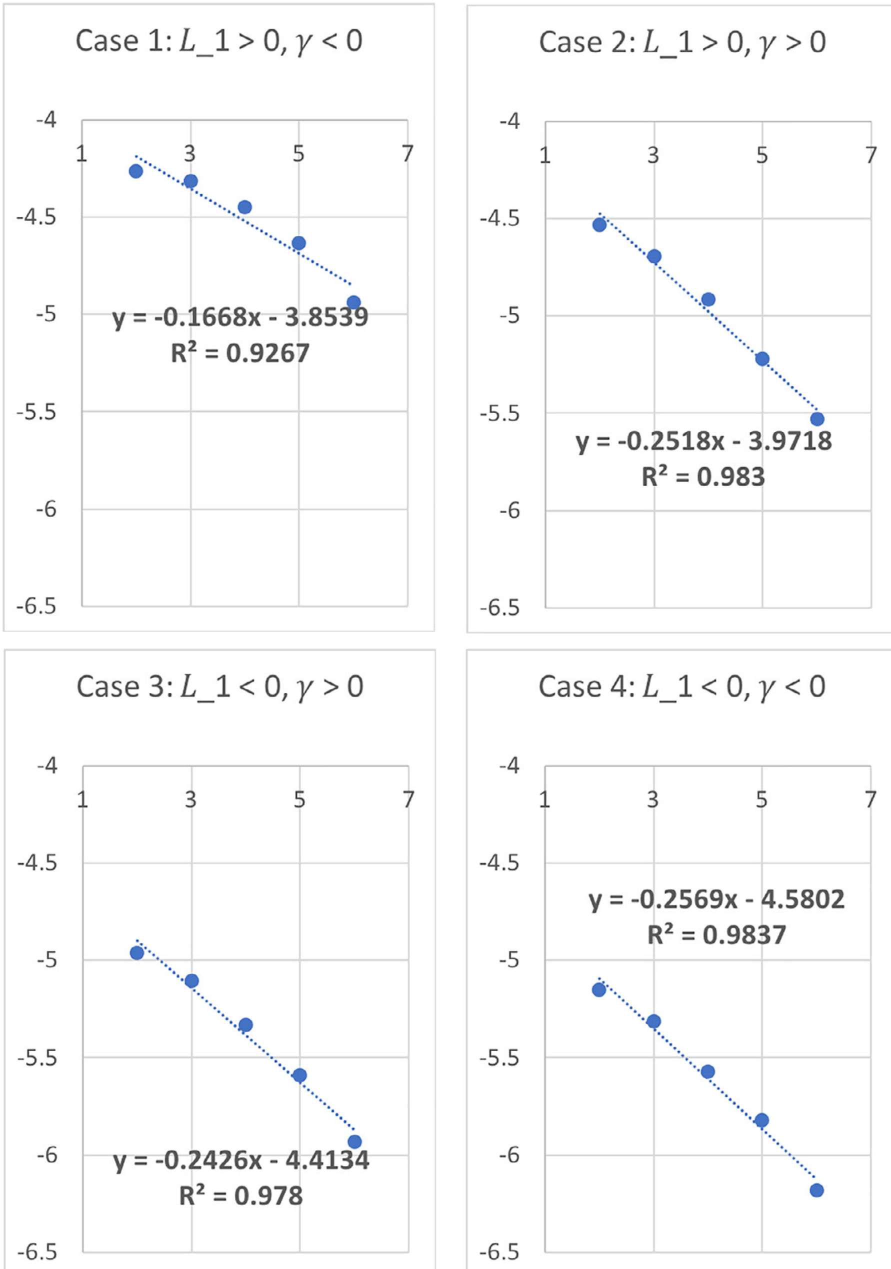

Figure 1. Values of $\log_2(me(l))$ for $l = 2, 3, 4, 5, 6$.

If $\hat{X}^{(l)}$ converges at the rate of order $\beta \in (0, +\infty)$ in L^1-norm, then there exists a constant $\beta > 0$ such that $2^{\beta l}\mathbb{E}[|X_1 - X_1^{(l)}|] = O(1)$, implying that $2^{\beta l}\mathbb{E}[|\hat{X}_1^{(l+1)} - \hat{X}_1^{(l)}|] = O(1)$ and vice-versa. In this case, we can write $\log_2 me(l) = -\beta l + C + o(1)$, for some constant $C \in \mathbb{R}$. Thus β can be estimated by the regression method.

Figure 1 shows the simulation result of $\log_2 me(l)$ for $l = 2, \ldots, 6$. We draw the regression lines to estimate the empirical rates of convergence β in each case. In Case 1, the empirical rate of convergence, which is 0.1668, is almost the same as the theoretical rate, which is 1/6. In the other cases, the empirical rates are slightly better than the theoretical rate.

Note that in Case 4, the tamed-adaptive Euler-Maruyama approximation converges in infinitive time intervals while in other cases, it converges in any finite time intervals.

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