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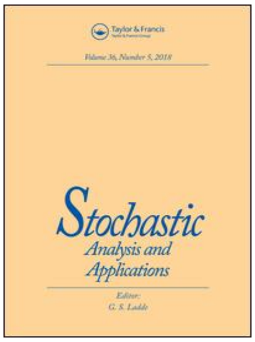

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Tamed-adaptive Euler-Maruyama approximation for SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients

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ABSTRACT

We propose a tamed-adaptive Euler-Maruyama approximation scheme for stochastic differential equations with locally Lipschitz continuous, polynomial growth drift, and locally Hölder continuous, polynomial growth diffusion coefficients. We consider the strong convergence and the stability of the new scheme. In particular, we show that under some sufficient conditions for the stability of the exact solution, the tamed-adaptive scheme converges strongly in both finite and infinite time intervals.

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Euler-Maruyama approximation; Hölder continuous diffusion; strong approximation; polynomial growth coefficient

1. Introduction

Throughout this paper, we consider the process $(X_t)_{t\geq 0}$ given by the following stochastic differential equation (SDE),

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad x_0 \in \mathbb{R}, t \geq 0, \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions; b and σ are real-valued measurable functions. Such SDEs arise in many areas of science and engineering, from population genetics to financial mathematics (see, e.g., Chapter 7 in [1], and [2]). Therefore, it is important to find effective methods to solve SDEs numerically. If coefficients b and σ are Lipschitz continuous, the numerical approximation for X has been well studied (see [1, 3]). However, numerical analysis for SDEs with non-Lipschitz coefficients is still a very active research area. In [4], Hutzenthaler et al. showed the divergence of the classical Euler-Maruyama approximation in L^p-norm when applying for some classes of SDEs with super-linear growth coefficients. After that, they introduced in [5] a tamed Euler-Maruyama approximation scheme and showed its strong convergence when applying for SDEs with locally Lipschitz continuous, super-linear growth drift coefficient and globally Lipschitz continuous diffusion coefficient. Since then, the tamed Euler-Maruyama has been developed by many authors for larger classes

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of SDEs with super-linear growth and low regularity coefficients (see, e.g., [6–10] and the references therein).

The strong convergence in infinite time of approximated solutions has attracted much attention recently. In [11], Fang and Giles introduced an adaptive Euler-Maruyama approximation. They showed that if b and σ satisfy the contractive Lipschitz condition (Assumption 9 in [11]), b is polynomial growth Lipschitz continuous, and σ is globally Lipschitz continuous, then the adaptive Euler-Maruyama approximation converges in L^p-norm in infinite time intervals. In [12], Li et al. studied the strong convergence in infinite time of a truncated Euler-Maruyama approximation scheme for SDEs with locally Lipschitz and polynomial growth coefficients. Their method is to use Khasminski's techniques on some well-designed Lyapunov functions.

The aim of this paper is to propose a numerical scheme that strongly converges in both finite and infinite time intervals for some class of one-dimensional SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients. By combining the tamed Euler-Maruyama approximation in [5, 9, 10] with the adaptive one in [11], we propose a tamed-adaptive Euler-Maruyama approximation scheme. We study the strong convergence of the scheme in both finite and infinite time intervals. The convergence rate to be established is optimal because it is similar to the standard results of the classical and tamed Euler-Maruyama schemes for SDEs with Hölder continuous diffusion coefficient (see [7, 8, 13, 14]). We also consider the stability of the tamed-adaptive scheme in L^p-norm where we obtain a similar result to Theorem 5 in [11] for a more general class of diffusion coefficients. The key idea of our argument is to use the Yamada-Watanabe approximation to obtain upper bounds for some p-th moments of the approximated solution. In comparison to [12], our method works for SDEs with locally Hölder continuous diffusion coefficient. However, a disadvantage point of using the Yamada-Watanabe approximation is that it can be applied only for one-dimensional SDEs while the method of Lyapunov functions in [11, 12] works for multi-dimensional SDEs.

The rest of this paper is organized as follows. The definition of tamed-adaptive Euler-Maruyama scheme and its convergence are established in Section 2. All the proofs are deferred to Section 3.

2. Main results

2.1. Assumptions

We consider the following assumptions on b and σ.

A1. There exist $\gamma \in \mathbb{R}, \eta \in [0, +\infty)$, and $p_0 \in [2, +\infty)$ such that

$$xb(x) + \frac{p_0 - 1}{2}|\sigma(x)|^2 \leq \gamma |x|^2 + \eta,$$

for any $x \in \mathbb{R}$.

A2. b is one-sided Lipschitz: there exists a constant L_1 such that

$$(x-y)(b(x)-b(y)) \leq L_1|x-y|^2,$$

for any $x, y \in \mathbb{R}$.

A3. b is locally Lipschitz continuous: there exist positive constants l and L_2 such that

$$|b(x)-b(y)| \leq L_2\left(1+|x|^l+|y|^l\right)|x-y|,$$

for any $x, y \in \mathbb{R}$.

A4. σ is $\left(\alpha+\frac{1}{2}\right)$-locally Hölder continuous: there exist positive constants m, L_3 and $\alpha \in [0, \frac{1}{2}]$ such that

$$|\sigma(x)-\sigma(y)| \leq L_3\left(1+|x|^m+|y|^m\right)|x-y|^{1/2+\alpha},$$

for any $x, y \in \mathbb{R}$.

It follows from Theorem 2.1 in [8] that if Assumptions **A1**, **A3**, **A4** hold for $p_0 \geq 4l+4,$ then Equation (1) has a unique strong solution. It is straighforward to verify that the result of that Theorem 2.1 also holds for $\alpha = 0$.

2.2. Tamed-adaptive Euler-Maruyama scheme

For each $\Delta \in (0,1),$ the tamed-adaptive Euler-Maruyama discretization of Equation (1) is defined as follows

$$\begin{cases} t_0 = 0, \quad \hat{X}_0 = x_0, \quad t_{k+1} = t_k + h_\Delta(\hat{X}_{t_k}), \\ \hat{X}_{t_{k+1}} = \hat{X}_{t_k} + b_\Delta(\hat{X}_{t_k})h_\Delta(\hat{X}_{t_k}) + \sigma_\Delta(\hat{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \end{cases} \tag{2}$$

where

$$h_\Delta(x) = \frac{\Delta}{(1+|b(x)|+|\sigma(x)|+|x|^l)^2}, \tag{3}$$

for some constant $l \geq 1,$ and b_Δ, σ_Δ are some approximations of b and σ which will be specified later.

The next result provides a sufficient condition for $t_k \to \infty$ as $k \to \infty,$ which implies that the tamed adaptive approximation scheme (2) is well-defined.

Proposition 2.1. *Suppose that there exist positive constants L and β such that coefficients b, σ, b_Δ and σ_Δ satisfy the following conditions*

T1. $|b(x)| \vee |\sigma(x)| \leq L(1+|x|^\beta)$;
T2. $x(b_\Delta(x) - b_\Delta(0)) \leq L|x|^2$;
T3. $|\sigma_\Delta(x)| \leq L|\sigma(x)|$ *and* $|b_\Delta(x)| \leq L|b(x)|$;
T4. $|\sigma_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$;

for any $x \in \mathbb{R}$. *Then*

$$\lim_{k \to +\infty} t_k = +\infty \quad \text{a.s.} \tag{4}$$

The proof is deferred to Section 3.3.

Under the assumptions of Proposition 2.1, we can define $\underline{t} := \max\{t_n : t_n \le t\}$ for the nearest time point before t, and $N_t := \max\{n : t_n \le t\}$ for the number of time steps approximation up to time t. Note that $\underline{t}$ is a stopping time. We define the piecewise constant interpolant process $\bar{X}_t = \hat{X}_{\underline{t}}$ and also define the standard continuous interpolant as

$$\begin{aligned} \hat{X}_t &= \hat{X}_{\underline{t}} + b_\Delta\left(\hat{X}_{\underline{t}}\right)(t - \underline{t}) + \sigma_\Delta\left(\hat{X}_{\underline{t}}\right)\left(W_t - W_{\underline{t}}\right) \\ &= \bar{X}_t + b_\Delta(\bar{X}_t)(t - \underline{t}) + \sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}}). \end{aligned} \tag{5}$$

Hence, $\hat{X}_t$ is the solution of the SDE

$$d\hat{X}_t = b_\Delta(\bar{X}_t)dt + \sigma_\Delta(\bar{X}_t)dW_t, \qquad \hat{X}_0 = x_0.$$

Remark 2.2. It is straightforward to verify that under Assumptions **A1–A4**, the following functions

$$b_\Delta(x) = b(x), \quad \sigma_\Delta(x) = \frac{\sigma(x)}{1 + \Delta^{1/2}|\sigma(x)|}. \tag{6}$$

satisfy all conditions of Proposition 2.1.

2.3. Moments

We first consider the moments of the exact solution. The following result should be known but we are not able to find it in any classical texts. The proof is given in Section 3.1.

Proposition 2.3. *Assume that coefficients b and σ satisfy the condition **A1**, and σ is bounded on every compact subset of $\mathbb{R}$. Then, for any $p \in [0, p_0]$,*

$$\mathbb{E}\left[|X_t|^p\right] \le \begin{cases} \left|x_0^2 e^{2\gamma t} + \frac{\eta}{\gamma}(e^{2\gamma t} - 1)\right|^{p/2} & \text{if } \gamma \ne 0, \\ |x_0^2 + 2\eta t|^{p/2} & \text{if } \gamma = 0. \end{cases} \tag{7}$$

We have the following bound on moments of the tamed-adaptive Euler-Maruyama approximation.

Theorem 2.4. *If coefficients $b, \sigma, b_\Delta, \sigma_\Delta$ satisfy Conditions **T1–T4**, and there exist $\gamma \in \mathbb{R}, \eta \in [0, +\infty]$ and $p_0 \in [2, +\infty)$ such that for all $x \in \mathbb{R}$ and $t \ge 0$, it holds*

$$x b_\Delta(x) + \frac{p_0 - 1}{2}\sigma_\Delta^2(x) \le \gamma|x|^2 + \eta. \tag{8}$$

Then, for any positive integer $k \le p_0/2$, there exists a positive constant $K = K(x_0, k, \eta, \gamma, L)$, which does not depend on t or Δ, such that

$$\mathbb{E}\left[|\hat{X}_t|^{2k}\right]\vee\mathbb{E}\left[|\bar{X}_t|^{2k}\right] \leq \begin{cases} Ke^{2k\gamma t} & \textit{if } \gamma > 0 \\ K(1+t)^k & \textit{if } \gamma = 0 \\ K & \textit{if } \gamma < 0 \end{cases}. \tag{9}$$

The proof of Theorem 2.4 is deferred to Section 3.4.

Remark 2.5. Under Assumptions **A1–A4**, functions b_Δ and σ_Δ defined in (6) satisfy conditions **T2–T4** and (8). If $\gamma < 0$, then the approximated solution is stable in the sense that for any $0 \leq p \leq 2[p_0/2]$ there exits a constant K, which does not depend on Δ, such that

$$\sup_{t\geq 0}\mathbb{E}\left[|\hat{X}_t|^p\right]\vee\mathbb{E}\left[|\bar{X}_t|^p\right] < K.$$

Here we use the notation $[p_0/2]$ for the integer part of $p_0/2$. Therefore, our result slightly improves Theorem 5 in [11] since we are able to relax the boundedness condition on σ. This improvement is due to the Yamada-Watanabe approximation.

Remark 2.6. Let N_T be the number of timesteps required by a path approximation on $[0, T]$ for any $T > 0$. Suppose that all conditions of Theorem 2.4 hold, then we have the following bound on the expectation of N_T,

$$\mathbb{E}\left[(N_T - 1)^p\right] \leq C(p)\Delta^{-p}, \tag{10}$$

for any $p \in \left[0, \frac{[p_0/2]}{\beta\vee l}\right]$, where $C(p)$ is a positive constant which does not depend on Δ.

By following the argument in the proof of Lemma 2 in [11], we can obtain the estimate (10) as a consequence of Theorem 2.4.

2.4. Convergence

Theorem 2.7. *Let Assumptions **A1–A4** hold and $p_0 \geq 2l\vee(2+4\alpha+4m)$. Suppose that functions b_Δ, σ_Δ satisfy all conditions of Theorem 2.4, and*

$$|b(x) - b_\Delta(x)| \leq L_4\Delta|b(x)|, \quad |\sigma(x) - \sigma_\Delta(x)| \leq L_4\Delta^{1/2}|\sigma(x)|^2, \tag{11}$$

for some constant $L_4 > 0$.

Then, for any $T > 0$, there exists a positive constant $C_T = C(x_0, L, L_1, L_2, L_3, L_4, \gamma, \eta, T)$ such that

$$\sup_{0\leq t\leq T}\mathbb{E}\left[|\hat{X}_t - X_t|\right] \leq \begin{cases} C_T\Delta^\alpha & \text{if } 0 < \alpha \leq \dfrac{1}{2}, \\ \dfrac{C_T}{\log\dfrac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{12}$$

Moreover, if $L_1 < 0$ and $\gamma < 0$, then there exists a positive constant $C = C(x_0, L, L_1, L_2, L_3, L_4, \gamma, \eta)$, which does not depend on T, such that

$$\sup_{t\geq 0}\mathbb{E}\left[|\hat{X}_t - X_t|\right] \leq \begin{cases} C\Delta^{\alpha} & \text{if } 0<\alpha\leq \frac{1}{2}, \\ \dfrac{C}{\log\frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{13}$$

The proof of Theorem 2.7 is deferred to Section 3.5.

Remark 2.8. It is straightforward to verify that under Assumptions **A1–A4**, the functions b_Δ and σ_Δ defined in (6) also satisfy condition (11).

3. Proofs

3.1. Proof of Proposition 2.3

Applying Itô's formula for $e^{-p\gamma t}|X_t|^p$ and the Condition **A1**, for any $p\in[2,p_0]$, we have

$$\begin{aligned} e^{-p\gamma t}|X_t|^p &= |x_0|^p + \int_0^t pe^{-p\gamma s}|X_s|^{p-2}\left[-\gamma|X_s|^2 + X_s b(X_s) + \frac{p-1}{2}\sigma^2(X_s)\right]ds \\ &\quad + \int_0^t pe^{-p\gamma s}|X_s|^{p-1}\sigma(X_s)dW_s \\ &\leq |x_0|^p + \int_0^t p\eta e^{-p\gamma s}|X_s|^{p-2}ds + \int_0^t pe^{-p\gamma s}|X_s|^{p-1}\sigma(X_s)dW_s. \end{aligned} \tag{14}$$

For each $N>0$, we consider $\tau_N = \inf\{t\geq 0 : |X_t|\geq N\}$. It follows from (14) that

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq |x_0|^p + \int_0^t p\eta\mathbb{E}\left[e^{-p\gamma(s\wedge\tau_N)}|X_{s\wedge\tau_N}|^{p-2}\right]ds.$$

Since $(p-2)|x|^p + 2\geq p|x|^{p-2}$, we get

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq |x_0|^p + 2\eta te^{p|\gamma|t} + (p-2)\eta\int_0^t \mathbb{E}\left[e^{-p\gamma(s\wedge\tau_N)}|X_{s\wedge\tau_N}|^p\right]ds.$$

It then follows from Gronwall's inequality that

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq \left(|x_0|^p + 2\eta te^{p|\gamma|t}\right)e^{(p-2)\eta t}.$$

This implies

$$\mathbb{P}[\tau_N < t] \leq \left(|x_0|^p + 2\eta te^{p|\gamma|t}\right)e^{(p-2)\eta t}e^{p|\gamma|t}N^{-p}.$$

Thus, $\tau_N\uparrow\infty$ as $N\to\infty$. It follows from (14) that

$$\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] \leq |x_0|^p + \mathbb{E}\left[\int_0^{t\wedge\tau_N} p\eta e^{-p\gamma s}|X_s|^{p-2}ds\right]. \tag{15}$$

Let $N\to\infty$ in (15), and using Fatou's lemma for the left hand side, and the monotone convergence theorem for the right hand side, we get

$$\mathbb{E}\left[e^{-p\gamma t}|X_t|^p\right] \le |x_0|^p + \int_0^t p\eta e^{-p\gamma s}\mathbb{E}\left[|X_s|^{p-2}\right]ds. \tag{16}$$

Let $p=2$, we get

$$\mathbb{E}\left[e^{-2\gamma t}|X_t|^2\right] \le |x_0|^2 + \int_0^t 2\eta e^{-2\gamma s}ds,$$

which implies (7) for $p=2$. Furthermore, thanks to Hölder's inequality, (7) holds for any $p\in[0,2]$.

Suppose that (7) holds for some $p=p_1\in[0,p_0-2]$, i.e.,

$$\mathbb{E}\left[|X_t|^{p_1}\right] \le \begin{cases} |x_0^2e^{2\gamma t}+\frac{\eta}{\gamma}(e^{2\gamma t}-1)|^{p_1/2} & \text{if } \gamma\neq 0,\\ |x_0^2+2\eta t|^{p_1/2} & \text{if } \gamma=0.\end{cases} \tag{17}$$

We shall show that (7) holds for $p=p_1+2$. Thanks to the estimate (16), we have

$$\mathbb{E}\left[e^{-(p_1+2)\gamma t}|X_t|^{p_1+2}\right] \le |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta e^{-(p_1+2)\gamma s}\mathbb{E}\left[|X_s|^{p_1}\right]ds.$$

<u>If $\gamma=0$:</u> using the estimate (17) to obtain

$$\begin{aligned}\mathbb{E}\left[|X_t|^{p_1+2}\right] &\le |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta|x_0^2+2\eta s|^{p_1/2}ds.\\ &= |x_0|^{p_1+2} + \left(|x_0^2+2\eta t|^{(p_1+2)/2} - |x_0^2|^{(p_1+2)/2}\right)\\ &= |x_0^2+2\eta t|^{(p_1+2)/2}.\end{aligned}$$

<u>If $\gamma\neq 0$:</u> using the estimate (17) to obtain

$$\begin{aligned}\mathbb{E}\left[e^{-(p_1+2)\gamma t}|X_t|^{p_1+2}\right] &\le |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta e^{-(p_1+2)\gamma s}\left|x_0^2e^{2\gamma s}+\frac{\eta}{\gamma}(e^{2\gamma s}-1)\right|^{p_1/2}ds.\\ &= |x_0|^{p_1+2} + \int_0^t (p_1+2)\eta e^{-2\gamma s}\left|x_0^2+\frac{\eta}{\gamma}-\frac{\eta}{\gamma}e^{-2\gamma s}\right|^{p_1/2}ds.\\ &= |x_0|^{p_1+2} + \int_0^t \frac{1}{2}(p_1+2)\left|x_0^2+\frac{\eta}{\gamma}-\frac{\eta}{\gamma}e^{-2\gamma s}\right|^{p_1/2}d\left(x_0^2+\frac{\eta}{\gamma}-\frac{\eta}{\gamma}e^{-2\gamma s}\right).\\ &= |x_0|^{p_1+2} + \left|x_0^2+\frac{\eta}{\gamma}-\frac{\eta}{\gamma}e^{-2\gamma t}\right|^{(p_1+2)/2} - |x_0^2|^{(p_1+2)/2}\\ &= \left|x_0^2+\frac{\eta}{\gamma}-\frac{\eta}{\gamma}e^{-2\gamma t}\right|^{(p_1+2)/2}.\end{aligned}$$

This implies

$$\mathbb{E}\left[|X_t|^{p_1+2}\right] \le \left|x_0^2e^{2\gamma t}+\frac{\eta}{\gamma}e^{2\gamma t}-\frac{\eta}{\gamma}\right|^{(p_1+2)/2} = \left|x_0^2e^{2\gamma t}+\frac{\eta}{\gamma}(e^{2\gamma t}-1)\right|^{(p_1+2)/2}.$$

By the induction principle, we obtain (7) for any $p\in[0,p_0]$.

3.2. Yamada and Watanabe approximation

We recall the approximation technique of Yamada and Watanabe (see [13, 15]). For each $\delta > 1$ and $\varepsilon > 0$ there exists a continuous function $\psi_{\delta\varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with $\operatorname{supp}\psi_{\delta\varepsilon} \subset [\varepsilon/\delta; \varepsilon]$ such that

$$\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta\varepsilon}(z)dz = 1; \quad 0 \le \psi_{\delta\varepsilon}(z) \le \frac{2}{z\log\delta}, \quad z > 0.$$

Define

$$\phi_{\delta\varepsilon}(x) := \int_0^{|x|}\int_0^{y} \psi_{\delta\varepsilon}(z)dzdy, \quad x \in \mathbb{R}.$$

It is easy to verify that $\phi_{\delta\epsilon}$ has the following useful properties: for any $x \in \mathbb{R}$,

YW1. $\phi'_{\delta\varepsilon}(x) = \frac{x}{|x|}\phi'_{\delta\varepsilon}(|x|)$,
YW2. $0 \le |\phi'_{\delta\varepsilon}(x)| \le 1$,
YW3. $|x| \le \varepsilon + \phi_{\delta\varepsilon}(x)$,
YW4. $\frac{\phi'_{\delta\varepsilon}(|x|)}{|x|} \le \frac{\delta}{\varepsilon}$,
YW5. $\phi''_{\delta\varepsilon}(|x|) = \psi_{\delta\varepsilon}(|x|) \le \frac{2}{|x|\log\delta}1_{[\frac{\varepsilon}{\delta};\varepsilon]}(|x|) \le \frac{2\delta}{\varepsilon\log\delta}$.

3.3. Proof of Proposition 2.1

Throughout the proof, we use the following result, which is a consequence of the strong Markov property of W,

$$\mathbb{E}\left[(W_s - W_{\underline{s}})^r|\mathcal{F}_{\underline{s}}\right] = \begin{cases} 0 & \text{if } r \text{ is an odd integer} \\ \alpha_r(s-\underline{s})^{r/2} & \text{if } r \text{ is an even integer,} \end{cases} \tag{18}$$

for some positive constant α_r.

Proof of Proposition 2.1. We also use the projection method like in [11]. However, to deal with the superlinear growth of σ, we use the Yamada-Watanabe function $\phi_{\delta\epsilon}$ instead of L^p-norm.

For each $H > |x_0|$, we define a projected approximation scheme as follows:

$$\begin{cases} t_0^H = 0, \quad t_{k+1}^H = t_k^H + h_\Delta(\hat{X}^H_{t_k^H}), \\ \hat{X}^H_{t_{k+1}^H} = P_H\left(\hat{X}^H_{t_k^H} + b_\Delta\left(\hat{X}^H_{t_k^H}\right)h_\Delta(\hat{X}^H_{t_k^H}) + \sigma_\Delta\left(\hat{X}^H_{t_k^H}\right)(W_{t_{k+1}^H} - W_{t_k^H})\right), \end{cases}$$

where $P_H(Y) \triangleq \min(1, H/|Y|)Y$ and therefore $|\hat{X}^H_{t_k^H}| \le H$ for all k. Thus $h_\Delta(\hat{X}^H_{t_k^H}) \ge C(H, L, l, m)\Delta$, which implies that $t_k^H \uparrow \infty$ as $k \to \infty$. We also note that for each k, t_k^H is a stopping time and t_{k+1}^H is $\mathcal{F}_{t_k^H}$-measurable. Set $\underline{t}^H = \max\{t_k^H : t_k^H \le t\}$. Then $\underline{t}^H$ is also a stopping time.

The piecewise constant approximation for intermediate times is again $\bar{X}_t^H = \hat{X}^H_{\underline{t}^H}$, and the continuous approximation is

$$\begin{aligned}\hat{X}_t^H &= P_H\Big(\hat{X}_{\underline{t}^H}{}^H + b_\Delta\Big(\hat{X}_{\underline{t}^H}{}^H\Big)(t-\underline{t}^H) + \sigma_\Delta\Big(\hat{X}_{\underline{t}^H}{}^H\Big)\big(W_t - W_{\underline{t}^H}\big)\Big)\\ &= P_H\Big(\bar{X}_t^H + b_\Delta\Big(\bar{X}_t^H\Big)(t-\underline{t}^H) + \sigma_\Delta\Big(\bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)\Big).\end{aligned}$$

Firstly, we note that $P_H(Y) \triangleq \min(1, H/|Y|)Y$ implies $\phi_{\delta\varepsilon}(P_H(Y)) \le \phi_{\delta\varepsilon}(Y)$. Hence,

$$\begin{aligned}\phi_{\delta\varepsilon}\Big(\hat{X}_t^H\Big) &= \phi_{\delta\varepsilon}\Big(P_H\Big(\bar{X}_t^H + b_\Delta\Big(\bar{X}_t^H\Big)(t-\underline{t}^H) + \sigma_\Delta\Big(\bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)\Big)\Big)\\ &\le \phi_{\delta\varepsilon}\Big(\bar{X}_t^H + b_\Delta\Big(\bar{X}_t^H\Big)(t-\underline{t}^H) + \sigma_\Delta\Big(\bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)\Big).\end{aligned}$$

Using Taylor's expansion, there exists an $(\mathcal{F}_t)$-adapted process (ξ_t) such that

$$\begin{aligned}\phi_{\delta\varepsilon}\Big(\hat{X}_t^H\Big) \le{}& \phi_{\delta\varepsilon}\Big(\bar{X}_t^H\Big) + \phi'_{\delta\varepsilon}\Big(\bar{X}_t^H\Big)\Big(b_\Delta\Big(\bar{X}_t^H\Big)(t-\underline{t}^H) + \sigma_\Delta\Big(\bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)\Big)\\ &+ \frac{1}{2}\phi''_{\delta\varepsilon}(\xi_t)\Big(b_\Delta\Big(\bar{X}_t^H\Big)(t-\underline{t}^H) + \sigma_\Delta\Big(\bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)\Big)^2.\end{aligned}$$

Note that $|\phi''_{\delta\varepsilon}(x)| \le \frac{2\delta}{\varepsilon\log\delta}$, thus

$$\begin{aligned}\phi_{\delta\varepsilon}\Big(\hat{X}_t^H\Big) \le{}& \phi_{\delta\varepsilon}\Big(\bar{X}_t^H\Big) + \phi'_{\delta\varepsilon}\Big(\bar{X}_t^H\Big)\Big[b_\Delta\Big(\bar{X}_t^H\Big) - b_\Delta(0)\Big](t-\underline{t}^H) + \phi'_{\delta\varepsilon}\Big(\bar{X}_t^H\Big)b_\Delta(0)(t-\underline{t}^H)\\ &+ \phi'_{\delta\varepsilon}\Big(\bar{X}_t^H\Big)\sigma_\Delta\Big(\bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)\\ &+ \frac{2\delta}{\varepsilon\log\delta}\Big(b_\Delta^2\Big(\bar{X}_t^H\Big)(t-\underline{t}^H)^2 + \sigma_\Delta^2\Big(\bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)^2\Big).\end{aligned}$$

Thanks to **T3**, we get

$$b_\Delta^2(\bar{X}_t^H)(t-\underline{t}^H) \le L^2 b^2\Big(\bar{X}_t^H\Big)\frac{\Delta}{b^2\Big(\bar{X}_t^H\Big)} = L^2\Delta.$$

It follows from **YW3** and **T2** that

$$\begin{aligned}\phi'_{\delta\varepsilon}\Big(\bar{X}_t^H\Big)\Big[b_\Delta\Big(\bar{X}_t^H\Big) - b_\Delta(0)\Big](t-\underline{t}^H) &= \frac{\phi'_{\delta\varepsilon}\Big(|\bar{X}_t^H|\Big)}{|\bar{X}_t^H|}\bar{X}_t^H\Big[b_\Delta\Big(\bar{X}_t^H\Big) - b_\Delta(0)\Big](t-\underline{t}^H)\\ &\le L|\bar{X}_t^H|\phi'_{\delta\varepsilon}\Big(|\bar{X}_t^H|\Big)(t-\underline{t}^H).\end{aligned}$$

Because $|x|\phi'_{\delta\varepsilon}(|x|) \le \phi_{\delta\varepsilon}(x) + \varepsilon$,

$$\phi'_{\delta\varepsilon}\Big(\bar{X}_t^H\Big)\Big[b_\Delta\Big(\bar{X}_t^H\Big) - b_\Delta(0)\Big](t-\underline{t}^H) \le L\Big(\phi_{\delta\varepsilon}\Big(\bar{X}_t^H\Big) + \varepsilon\Big)(t-\underline{t}^H).$$

Moreover, thanks to **T4** and the fact that $dW_t^2 = 2W_t dW_t + dt$, we have

$$\sigma_\Delta^2\Big(\underline{t}^H; \bar{X}_t^H\Big)\big(W_t - W_{\underline{t}^H}\big)^2 \le \frac{L^2}{\Delta}\left(2\int_{\underline{t}^H}^t \big(W_s - W_{\underline{s}^H}\big)dW_s + (t-\underline{t}^H)\right).$$

Hence,

$$\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \le \left[1 + L\left(t - \underline{t}^H\right)\right]\phi_{\delta\varepsilon}\left(\bar{X}_t^H\right) + \left(L\varepsilon + |b_\Delta(0)| + \frac{2L^2\delta\Delta}{\varepsilon\log\delta} + \frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)(t - \underline{t}^H) + \int_{\underline{t}^H}^t \phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right)\sigma_\Delta\left(\bar{X}_s^H\right)dW_s + \int_{\underline{t}^H}^t \frac{4L^2\delta}{\varepsilon\Delta\log\delta}\left(W_s - W_{\underline{s}^H}\right)dW_s. \tag{19}$$

Note that $e^{-Lt}(1 + L(t - \underline{t}^H)) \le e^{-L\underline{t}^H}$. By multiplying e^{-Lt} to both sides of (19), we have

$$e^{-Lt}\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \le M_t + e^{-L\underline{t}^H}\phi_{\delta\varepsilon}\left(\bar{X}_t^H\right) + \left(L\varepsilon + |b_\Delta(0)| + \frac{2L^2\delta\Delta}{\varepsilon\log\delta} + \frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)e^{-Lt}(t - \underline{t}^H) + \left[e^{-Lt} - e^{-L\left(\underline{t}^H + h\left(\bar{X}_t^H\right)\right)}\right]\left(\phi'_{\delta\varepsilon}\left(\bar{X}_t^H\right)\sigma_\Delta\left(\bar{X}_t^H\right)(W_t - W_{\underline{t}^H}) + \frac{4L^2\delta}{\varepsilon\log\delta\Delta}\left[(W_t - W_{\underline{t}^H})^2 - (t - \underline{t}^H)\right]\right), \tag{20}$$

where

$$M_t = \int_{\underline{t}^H}^t e^{-L\left(\underline{s}^H + h\left(\bar{X}_s^H\right)\right)}\left(\phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right)\sigma_\Delta\left(\bar{X}_s^H\right) + \frac{4L^2\delta}{\varepsilon\Delta\log\delta}\left(W_s - W_{\underline{s}^H}\right)\right)dW_s.$$

Let $\omega(t, \Delta)$ be the modulus of continuity of W, i.e.,

$$\omega(t, \Delta) = \sup_{s_1, s_2 \in [0, t]; |s_2 - s_1| \le \Delta} |W(s_1) - W(s_2)|.$$

For any $p > 0$, it follows from Theorem 1 in [16] that

$$\mathbb{E}\left[|\omega(t, \Delta)|^p\right] \le C_p(\Delta\log\frac{2t}{\Delta})^{p/2}. \tag{21}$$

Therefore,

$$e^{-Lt}\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \le M_t + e^{-L\underline{t}^H}\phi_{\delta\varepsilon}\left(\bar{X}_t^H\right) + \left(L\varepsilon + |b_\Delta(0)| + \frac{2L^2\delta\Delta}{\varepsilon\log\delta} + \frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)\int_{\underline{t}^H}^t e^{-Ls}ds + Le^{-L(t-\Delta)}(t - \underline{t}^H)\left(\frac{L}{\sqrt{\Delta}}\omega(t, \Delta) + \frac{4L^2\delta}{\varepsilon\Delta\log\delta}\omega(t, \Delta)^2\right). \tag{22}$$

It also follows from (20) that

$$e^{-Lt_{k+1}^H}\phi_{\delta\varepsilon}\left(\hat{X}_{t_k^H}\right) \le M_{t_k^H} + e^{-Lt_k^H}\phi_{\delta\varepsilon}\left(\hat{X}_{t_k^H}^H\right) + \left(L\varepsilon + |b_\Delta(0)| + \frac{2L^2\delta\Delta}{\varepsilon\log\delta} + \frac{2L^2\delta}{\varepsilon\Delta\log\delta}\right)\int_{t_k^H}^{t_{k+1}^H} e^{-Ls}ds, \tag{23}$$

where

$$M_{t_k^H} = \int_{t_k^H}^{t_{k+1}^H} e^{-L\left(\underline{s}^H + h\left(\bar{X}_s^H\right)\right)} \left(\phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right)\sigma_\Delta\left(\bar{X}_s^H\right) + \frac{4L^2\delta}{\varepsilon \log \delta\Delta}\left(W_s - W_{\underline{s}^H}\right)\right) dW_s.$$

Summing (23) over multiple timesteps and then adding (22) gives

$$\begin{aligned} e^{-Lt}|\hat{X}_t^H| &\le \varepsilon + e^{-Lt}\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) \\ &\le \varepsilon + \bar{M}_t + \phi_{\delta\varepsilon}(x_0) + \left(L\varepsilon + |b_\Delta(0)| + \frac{2L^2\delta\Delta}{\varepsilon \log \delta} + \frac{2L^2\delta}{\varepsilon\Delta \log \delta}\right)\int_0^t e^{-Ls} ds \\ &\quad + Le^{-L(t-\Delta)}(t - \underline{t}^H)\left(\frac{L}{\sqrt{\Delta}}\omega(t, \Delta) + \frac{4L^2\delta}{\varepsilon\Delta \log \delta}\omega(t, \Delta)^2\right), \end{aligned}$$

where

$$\bar{M}_t = \int_0^t e^{-L\left(\underline{s}^H + h\left(\bar{X}_s^H\right)\right)} \left(\phi'_{\delta\varepsilon}\left(\bar{X}_s^H\right)\sigma_\Delta\left(\bar{X}_s^H\right) + \frac{4L^2\delta}{\varepsilon \log \delta\Delta}\left(W_s - W_{\underline{s}^H}\right)\right) dW_s.$$

Hence, for any the stopping time $\tau \le t$, it follows from (21) that there exists a constant $C = C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))$, which does not depend on H such that

$$\mathbb{E}\left[e^{-L\tau}|\hat{X}_\tau^H|\right] \le C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))\left(1 + |\log t|\right).$$

Applying Proposition IV.4.7 in [17], for any $p \in (0, 1)$, we have

$$\mathbb{E}\left[\sup_{0\le s\le t} e^{-Lps}|\hat{X}_\tau^H|^p\right] \le \frac{2-p}{1-p} C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))^p\left(1 + |\log t|\right)^p.$$

Hence,

$$\mathbb{E}\left[\sup_{0\le s\le t} |\hat{X}_\tau^H|^p\right] \le \frac{2-p}{1-p} C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))^p\left(1 + |\log t|\right)^p e^{Lpt}. \tag{24}$$

On the other hand, for any $T > 0$,

$$\mathbb{P}[t_k < T] = \mathbb{P}\left[t_k < T, \sup_{0\le s\le T} |\hat{X}_s^H| > \frac{H}{2}\right] + \mathbb{P}\left[t_k < T, \sup_{0\le s\le T} |\hat{X}_s^H| \le \frac{H}{2}\right].$$

It follows from Markov's inequality and (24) that

$$\begin{aligned} \mathbb{P}\left[t_k < T, \sup_{0\le s\le T} |\hat{X}_s^H| > \frac{H}{2}\right] &\le \mathbb{P}\left[\sup_{0\le s\le T} |\hat{X}_s^H| > \frac{H}{2}\right] \le \left(\frac{2}{H}\right)^p \mathbb{E}\left[\sup_{0\le s\le t} |\hat{X}_\tau^H|^p\right] \\ &\le \left(\frac{2}{H}\right)^p \frac{2-p}{1-p} C(x_0, \varepsilon, \delta, \Delta, L, b_\Delta(0))^p\left(1 + |\log T|\right)^p e^{LpT}. \end{aligned}$$

On the other hand, on the set $\left\{\sup_{0\le s\le T} |\hat{X}_s^H| \le \frac{H}{2}\right\}, \hat{X}_s^H = \hat{X}_s$ for all $s \le T$, which, in turn, implies that $t_k^H = t_k$ if $t_k < T$. Hence,

$$\limsup_{k\to\infty}\mathbb{P}\left[t_k < T,\ \sup_{0\le s\le T}|\hat{X}_s^H| \le \frac{H}{2}\right] \le \limsup_{k\to\infty}\mathbb{P}\left[t_k^H < T\right] = 0,$$

which implies that

$$\limsup_{k\to\infty}\mathbb{P}[t_k < T] \le \left(\frac{2}{H}\right)^p \frac{2-p}{1-p} C(x_0,\varepsilon,\delta,\Delta,L,b_\Delta(0))^p \left(1+|\log T|\right)^p e^{LpT},$$

for any $H>0$. Let $H\to\infty$, we get $\limsup_{k\to\infty}\mathbb{P}[t_k < T] = 0$ for any $T>0$. This implies that $\lim_{k\to\infty} t_k = +\infty$ almost surely, which is the desired result.

3.4. Proof of Theorem 2.4

We start will the following key estimate on moments of $\hat{X}$.

Lemma 3.1. *Assume that Conditions **T1–T4** hold. Then for any $p>0$ and $T>0$, there exists a positive constant $C(p,L,T,x_0,\Delta)<\infty$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\hat{X}_t|^p\right] \le C(p,L,T,x_0,\Delta).$$

Proof. Applying Itô's formula for $e^{-Lt}\phi_{\delta\varepsilon}(\hat{X}_t)$ gives

$$\begin{aligned}
e^{-Lt}|\hat{X}_t| &\le e^{-Lt}\varepsilon + e^{-Lt}\phi_{\delta\varepsilon}(\hat{X}_t)\\
&\le \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\left[-L\phi_{\delta\varepsilon}(\hat{X}_s) + \phi'_{\delta\varepsilon}(\hat{X}_s)b_\Delta(\bar{X}_s) + \frac{1}{2}\phi''_{\delta\varepsilon}(\hat{X}_s)|\sigma_\Delta(\bar{X}_s)|^2\right]ds\\
&\quad + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\hat{X}_s)\sigma_\Delta(\bar{X}_s)dW_s.
\end{aligned}$$

Applying Taylor's expansion for $\phi'_{\delta\varepsilon}$, there exists an $(\mathcal{F}_s)$-adapted process $\xi = (\xi_s)$ such that

$$\begin{aligned}
\phi'_{\delta\varepsilon}(\hat{X}_s)b_\Delta(\bar{X}_s) &= \Big(\phi'_{\delta\varepsilon}(\bar{X}_s) + \phi''_{\delta\varepsilon}(\xi_s)\big(\hat{X}_s - \bar{X}_s\big)\Big)b_\Delta(\bar{X}_s)\\
&= \phi'_{\delta\varepsilon}(\bar{X}_s)b_\Delta(\bar{X}_s) + \phi''_{\delta\varepsilon}(\xi_s)\Big(b_\Delta(\bar{X}_s)(s-\underline{s}) + \sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\Big)b_\Delta(\bar{X}_s)\\
&= \phi'_{\delta\varepsilon}(\bar{X}_s)\big(b_\Delta(\bar{X}_s) - b_\Delta(0)\big) + \phi'_{\delta\varepsilon}(\bar{X}_s)b_\Delta(0)\\
&\quad + \phi''_{\delta\varepsilon}(\xi_s)\Big(b_\Delta^2(\bar{X}_s)(s-\underline{s}) + b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\Big)\\
&= \frac{\phi'_{\delta\varepsilon}(|\bar{X}_s|)}{|\bar{X}_s|}\bar{X}_s\big(b_\Delta(\bar{X}_s) - b_\Delta(0)\big) + \phi'_{\delta\varepsilon}(\bar{X}_s)b_\Delta(0)\\
&\quad + \phi''_{\delta\varepsilon}(\xi_s)\Big(b_\Delta^2(\bar{X}_s)(s-\underline{s}) + b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\Big)\\
&\le L|\bar{X}_s| + |b_\Delta(0)| + \frac{2\delta}{\varepsilon\log\delta}C\Delta + \frac{2\delta}{\varepsilon\log\delta}|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|.
\end{aligned}$$

Hence,

$$e^{-Lt}|\hat{X}_t| \le \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\Big[- L\phi_{\delta\varepsilon}(\hat{X}_s) + L|\bar{X}_s| + |b_\Delta(0)| + \frac{2\delta}{\varepsilon\log\delta}C\Delta + \frac{2\delta}{\varepsilon\log\delta}|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})| + \frac{\delta}{\varepsilon\log\delta}\frac{C}{\Delta}\Big]ds + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\hat{X}_s)\sigma_\Delta(\bar{X}_s)dW_s.$$

Note that

$$\begin{aligned} -\phi_{\delta\varepsilon}(\hat{X}_s) + |\bar{X}_s| &\le -|\hat{X}_s| + \varepsilon + |\bar{X}_s| \\ &\le \varepsilon + |b_\Delta(\bar{X}_s)(s - \underline{s})| + |\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|. \end{aligned}$$

Thus,

$$\begin{aligned} e^{-Lt}|\hat{X}_t| \le{}& (1 + Lt)\varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t Le^{-Ls}\Big(|b_\Delta(\bar{X}_s)(s - \underline{s})| + |\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|\Big)ds \\ &+ \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\hat{X}_s)\sigma_\Delta(\bar{X}_s)dW_s \\ &+ \int_0^t e^{-Ls}\Big[|b_\Delta(0)| + \frac{2\delta}{\varepsilon\log\delta}C\Delta + \frac{2\delta}{\varepsilon\log\delta}|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})| + \frac{\delta}{\varepsilon\log\delta}\frac{C}{\Delta}\Big]ds. \end{aligned} \tag{25}$$

Note that, for any $p > 0$, there exists a constant $C(p) > 0$ such that

$$\begin{aligned} &\mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|^p\Big] \\ &= \mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)|^p\mathbb{E}\Big[|W_s - W_{\underline{s}}|^p|\mathcal{F}_{\underline{s}}\Big]\Big] \\ &\le C(p)\mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)|^p(s - \underline{s})^{p/2}\Big]. \end{aligned}$$

Thanks to (3) and Condition **T4**, we have

$$\mathbb{E}\Big[|b_\Delta(\bar{X}_s)\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})|^p\Big] \le C(p, L).$$

Therefore, by choosing $\varepsilon = 1, \delta = 2$ in (25), it follows from **T1–T4**, Hölder's inequality and Burkholder-Davis-Gundy's inequality that for any $T > 0$, there exists a positive constant $C(p, L, T, x_0, \Delta) < \infty$ such that

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\hat{X}_t|^p\Big] \le C(p, L, T, x_0, \Delta).$$

□

Proof of Theorem 2.4. Thanks to Hölder's inequality, it is sufficient to show (8) for k is a positive interger and $k \le p_0/2$. We will use the induction method. Firstly, for $k = 1$, applying Itô's formula for $e^{-2\gamma t}\hat{X}_t^2$, we have

$$e^{-2\gamma t}\hat{X}_t^2 = x_0^2 + \int_0^t e^{-2\gamma s}\Big(-2\gamma\hat{X}_s^2 + 2\hat{X}_s b_\Delta(\bar{X}_s) + \sigma_\Delta^2(\bar{X}_s)\Big)ds + \int_0^t 2e^{-2\gamma s}\hat{X}_s\sigma_\Delta(\bar{X}_s)dW_s. \tag{26}$$

On the other hand, it follows from (5) that

$$\hat{X}_s^2 = \bar{X}_s^2 + 2\bar{X}_s b_\Delta(\bar{X}_s)(s-\underline{s}) + b_\Delta^2(\bar{X}_s)(s-\underline{s})^2 + \sigma_\Delta^2(\bar{X}_s)(W_s - W_{\underline{s}})^2$$
$$+ 2(\bar{X}_s + b_\Delta(\bar{X}_s)(s-\underline{s}))\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}}).$$

Thanks to **T3**, (18), and (3),

$$\max\left\{|\bar{X}_s b_\Delta(\bar{X}_s)(s-\underline{s})|, b_\Delta^2(\bar{X}_s)(s-\underline{s})^2, \mathbb{E}\left[\sigma_\Delta^2(\bar{X}_s)(W_s - W_{\underline{s}})^2|\mathcal{F}_{\underline{s}}\right]\right\} \le C\Delta. \tag{27}$$

Therefore,

$$\mathbb{E}\left[-2\gamma\hat{X}_s^2\right] \le \mathbb{E}\left[-2\gamma\bar{X}_s^2\right] + C|\gamma|\Delta. \tag{28}$$

A similar argument yields

$$\mathbb{E}\left[2\hat{X}_s b_\Delta(\bar{X}_s)\right] \le \mathbb{E}\left[2\bar{X}_s b_\Delta(\bar{X}_s)\right] + C|\gamma|\Delta. \tag{29}$$

It then follows from (8), (26), (28), (29), that

$$\mathbb{E}\left[e^{-2\gamma t}\hat{X}_t^2\right] \le |x_0|^2 + C(\eta + |\gamma|\Delta)\int_0^t e^{-2\gamma s}ds. \tag{30}$$

Note that $\bar{X}_t = \hat{X}_t - b_\Delta(\bar{X}_t)(t-\underline{t}) - \sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})$, which together with (27) implies the following estimate for any $p > 0$,

$$\begin{aligned}\mathbb{E}\left[|\bar{X}_t|^p\right] &\le 3^{p-1}\left(\mathbb{E}\left[|\hat{X}_t|^p\right] + \mathbb{E}\left[|b_\Delta(\bar{X}_t)(t-\underline{t})|^p\right] + \mathbb{E}\left[|\sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})|^p\right]\right)\\ &\le 3^{p-1}\left(\mathbb{E}\left[|\hat{X}_t|^p\right] + C\Delta^p + C\Delta^{p/2}\right).\end{aligned} \tag{31}$$

It follows from (30) and (31) that (9) holds for $k=1$.

Secondly, we assume that (9) holds for any $k \le k_0 \le [p_0/2] - 1$, we will show that (9) still holds for $k = k_0 + 1$.

By applying Itô's formula for $e^{-p\gamma s}\hat{X}_t^p$ with $p = 2(k_0+1)$ being an even integer, we have

$$\begin{aligned}e^{-p\gamma t}|\hat{X}_t|^p = |x_0|^p &+ \int_0^t pe^{-p\gamma s}\left(-\gamma|\hat{X}_s|^p + \hat{X}_s^{p-1}b_\Delta(\bar{X}_s) + \frac{p-1}{2}|\hat{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right)ds\\ &+ \int_0^t pe^{-p\gamma s}|\hat{X}_s|^{p-2}\hat{X}_s\sigma_\Delta(\bar{X}_s)dW_s.\end{aligned} \tag{32}$$

It follows from (5) and the Newton expansion formula that for any positive integer q,

$$|\hat{X}_s|^q = \sum_{0\le i,j,r\le q, i+j+r=q} \frac{q!}{i!j!r!}(\bar{X}_s)^i\left(b_\Delta(\bar{X}_s)(s-\underline{s})\right)^j\left(\sigma_\Delta(\bar{X}_s)(W_s - W_{\underline{s}})\right)^r. \tag{33}$$

Thanks to (18), we have

$$\begin{aligned}\mathbb{E}\left[-\gamma|\hat{X}_s|^p|\mathcal{F}_{\underline{s}}\right] &= -\gamma|\bar{X}_s|^p - p\gamma\bar{X}_s b_\Delta(\bar{X}_s)(s-\underline{s})|\bar{X}_s|^{p-2}\\ &\quad - \sum_{0\le i\le p-2, i+j+2r=p}\frac{\gamma q!}{i!j!k!}(\bar{X}_s)^i\left(b_\Delta(\bar{X}_s)(s-\underline{s})\right)^j\left(\sigma_\Delta^2(\bar{X}_s)(s-\underline{s})\right)^r\alpha_{2r}.\end{aligned}$$

Thanks to (27), we get

$$\mathbb{E}\left[-\gamma|\hat{X}_s|^p\right] \leq \mathbb{E}\left[-\gamma|\bar{X}_s|^p\right] + C|\gamma|\sum_{i=0}^{p-2}\mathbb{E}\left[|\bar{X}_s|^i\right]. \tag{34}$$

Choose $q = p-1$ and $q = p-2$ in (33), by the same argument, we also have

$$\mathbb{E}\left[\hat{X}_s^{p-1}b_\Delta(\bar{X}_s)\right] \leq \bar{X}_s^{p-1}b_\Delta(\bar{X}_s) + C\sum_{i=0}^{p-2}\mathbb{E}\left[|\bar{X}_s|^i\right] \tag{35}$$

and

$$\mathbb{E}\left[\frac{p-1}{2}|\hat{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right] \leq \frac{p-1}{2}|\bar{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s) + C\sum_{i=0}^{p-2}\mathbb{E}\left[|\bar{X}_s|^i\right]. \tag{36}$$

Combining (34)–(36) to get

$$\begin{aligned}
&\mathbb{E}\left[-\gamma|\hat{X}_s|^p + \hat{X}_s^{p-1}b_\Delta(\bar{X}_s) + \frac{p-1}{2}|\hat{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right] \\
&\quad \leq \mathbb{E}\left[-\gamma|\bar{X}_s|^p + \bar{X}_s^{p-1}b_\Delta(\bar{X}_s) + \frac{p-1}{2}|\bar{X}_s|^{p-2}\sigma_\Delta^2(\bar{X}_s)\right] + C\sum_{i=0}^{p-2}\mathbb{E}\left[|\bar{X}_s|^i\right] \\
&\quad \leq C\sum_{i=0}^{p-2}\mathbb{E}\left[|\bar{X}_s|^i\right].
\end{aligned} \tag{37}$$

From (32), (37), (31) and the inductive assumption, we obtain that (9) holds for $k = k_0 + 1,$ which implies the desired result.

3.5. Proof of Theorem 2.7

We need the following uniformly in time bound for the difference between $\hat{X}$ and $\bar{X}$.

Lemma 3.2. *Suppose that coefficients b_Δ, σ_Δ satisfy all conditions of Theorem 2.4, then there exists a positive constant $C_p = C(p, L)$ such that*

$$\sup_{t\geq 0}\mathbb{E}\left[|\hat{X}_t - \bar{X}_t|^p\right] \leq C_p\Delta^{p/2}, \tag{38}$$

for any $p \geq 0$.

Proof. From (5),

$$\begin{aligned}
|\hat{X}_t - \bar{X}_t|^p &= |b_\Delta(\bar{X}_t)(t-\underline{t}) + \sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})|^p \\
&\leq 2^{p-1}\left(|b_\Delta(\bar{X}_t)(t-\underline{t})|^p + |\sigma_\Delta(\bar{X}_t)(W_t - W_{\underline{t}})|^p\right) \\
&\leq 2^{p-1}\left(|b_\Delta(\bar{X}_t)|^p|h_\Delta(\bar{X}_t)|^p + |\sigma_\Delta(\bar{X}_t)|^p|W_t - W_{\underline{t}}|^p\right).
\end{aligned}$$

By applying **T3** and (3), we have

$$|b_\Delta(\bar{X}_t)h_\Delta(\bar{X}_t)| \le \frac{L\Delta}{4} \quad \text{and} \quad |\sigma_\Delta(\bar{X}_t)|h_\Delta(\bar{X}_t)|^{1/2}| \le L\Delta^{1/2}.$$

Using (18), we obtain the desired result. □

Proof of Theorem 2.7. Put $Y_t = X_t - \hat{X}_t$. Applying the property **YW3** and Itô's formula for $e^{-L_1 t}\phi_{\delta\varepsilon}(Y_t)$ gives

$$\begin{aligned}
e^{-L_1 t}|Y_t| &\le e^{-L_1 t}\varepsilon + e^{-L_1 t}\phi_{\delta\varepsilon}(Y_t) \\
&= e^{-L_1 t}\varepsilon + \int_0^t e^{-L_1 s}\Big[-L_1\phi_{\delta\varepsilon}(Y_s) + \phi'_{\delta\varepsilon}(Y_s)\big(b(X_s) - b_\Delta(\bar{X}_s)\big) + \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)|\sigma(X_s) - \sigma_\Delta(\bar{X}_s)|^2\Big]ds \\
&+ \int_0^t e^{-L_1 s}\phi'_{\delta\varepsilon}(Y_s)\big(\sigma(X_s) - \sigma_\Delta(\bar{X}_s)\big)dW_s.
\end{aligned} \tag{39}$$

Set $J_1(s) = \phi'_{\delta\varepsilon}(Y_s)(b(X_s) - b_\Delta(\bar{X}_s))$ and $J_2(s) = \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)|\sigma(X_s) - \sigma_\Delta(\bar{X}_s)|^2$. Firstly, we write

$$J_1(s) = \phi'_{\delta\varepsilon}(Y_s)\big(b(X_s) - b(\hat{X}_s)\big) + \phi'_{\delta\varepsilon}(Y_s)\big(b(\hat{X}_s) - b(\bar{X}_s)\big) + \phi'_{\delta\varepsilon}(Y_s)\big(b(\bar{X}_s) - b_\Delta(\bar{X}_s)\big).$$

Thanks to properties **YW1**, **YW2**, assumptions **A2**, **A3** and (11), we have

$$\begin{aligned}
J_1(s) &\le \frac{\phi'_{\delta\varepsilon}(|Y_s|)}{|Y_s|}Y_s\big(b(X_s) - b(\hat{X}_s)\big) + |\phi'_{\delta\varepsilon}(Y_s)\big(b(\hat{X}_s) - b(\bar{X}_s)\big)| + |\phi'_{\delta\varepsilon}(Y_s)\big(b(\bar{X}_s) - b_\Delta(\bar{X}_s)\big)| \\
&\le L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + L_2\Big(1 + |\hat{X}_s|^l + |\bar{X}_s|^l\Big)|\hat{X}_s - \bar{X}_s| + C\Delta\Big(1 + |\bar{X}_s|^{l+1}\Big) \\
&\le L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + \frac{3}{2}L_2\Delta^{1/2}\Big(1 + |\hat{X}_s|^{2l} + |\bar{X}_s|^{2l}\Big) + \frac{1}{2}L_2\Delta^{-1/2}|\hat{X}_s - \bar{X}_s|^2 + C\Delta\Big(1 + |\bar{X}_s|^{l+1}\Big).
\end{aligned} \tag{40}$$

Secondly, we write

$$J_2(s) = \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)|\sigma(X_s) - \sigma(\hat{X}_s) + \sigma(\hat{X}_s) - \sigma(\bar{X}_s) + \sigma(\bar{X}_s) - \sigma_\Delta(\bar{X}_s)|^2.$$

By using the property **YW5**, the assumption **A4** and (11), we have

$$
\begin{aligned}
J_2(s) &\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\big(|\sigma(X_s)-\sigma(\hat{X}_s)|^2+|\sigma(\hat{X}_s)-\sigma(\bar{X}_s)|^2+|\sigma(\bar{X}_s)-\sigma_\Delta(\bar{X}_s)|^2\big)\\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)(L_3^2\big(1+|X_s|^m+|\hat{X}_s|^m\big)^2|X_s-\hat{X}_s|^{1+2\alpha}+\\
&\quad + L_3^2\big(1+|\hat{X}_s|^m+|\bar{X}_s|^m\big)^2|\hat{X}_s-\bar{X}_s|^{1+2\alpha}+L_4\Delta|\sigma(\bar{X}_s)|^4)\\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)(3L_3^2\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)|X_s-\hat{X}_s|^{1+2\alpha}+\\
&\quad + 3L_3^2\big(1+|\hat{X}_s|^{2m}+|\bar{X}_s|^{2m}\big)|\hat{X}_s-\bar{X}_s|^{1+2\alpha}+L_4\Delta|\sigma(\bar{X}_s)|^4)\\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)+\frac{9L_3^2\delta}{\varepsilon\log\delta}\big(1+|\hat{X}_s|^{2m}+|\bar{X}_s|^{2m}\big)|\hat{X}_s-\bar{X}_s|^{2\alpha+1}\\
&\quad +\frac{3L_4\delta\Delta|\sigma(\bar{X}_s)|^4}{\varepsilon\log\delta}\\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\big(1+|\hat{X}_s|^{2m}+|\bar{X}_s|^{2m}\big)^2\Delta^{1/2+\alpha}\\
&\quad +\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-1/2-\alpha}|\hat{X}_s-\bar{X}_s|^{4\alpha+2}+\frac{C\delta\Delta\big(|\bar{X}_s|^{2+4\alpha+4m}+1\big)}{\varepsilon\log\delta}\\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+|X_s|^{2m}+|\hat{X}_s|^{2m}\big)+\frac{27L_3^2\delta}{2\varepsilon\log\delta}\big(1+|\hat{X}_s|^{4m}+|\bar{X}_s|^{4m}\big)\Delta^{1/2+\alpha}\\
&\quad +\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-1/2-\alpha}|\hat{X}_s-\bar{X}_s|^{2+4\alpha}+\frac{C\delta\Delta\big(|\bar{X}_s|^{2+4\alpha+4m}+1\big)}{\varepsilon\log\delta}.
\end{aligned}
\tag{41}
$$

A combination of (39), (40), (41) and the property $-L_1\phi_{\delta\varepsilon}(x)+L_1\phi'_{\delta\varepsilon}(|x|)|x|\le \max\{L_1\varepsilon;0\}$ implies

$$
\begin{aligned}
&\mathbb{E}\big[e^{-L_1t}|Y_t|\big]\\
&\le e^{-L_1t}\varepsilon+\int_0^t e^{-L_1s}[\max\{L_1\varepsilon;0\}+\frac{3}{2}L_2\Delta^{1/2}\Big(1+\mathbb{E}\Big[|\hat{X}_s|^{2l}\Big]+\mathbb{E}\Big[|\bar{X}_s|^{2l}\Big]\Big)\\
&+\frac{1}{2}L_2\Delta^{-1/2}\mathbb{E}\big[|\hat{X}_s-\bar{X}_s|^2\big]+C\Delta\Big(1+\mathbb{E}\Big[|\bar{X}_s|^{l+1}\Big]\Big)+\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\big(1+\mathbb{E}\big[|X_s|^{2m}\big]+\mathbb{E}\big[|\hat{X}_s|^{2m}\big]\big)\\
&+\frac{27L_3^2\delta}{2\varepsilon\log\delta}\big(1+\mathbb{E}\big[|\hat{X}_s|^{4m}\big]+\mathbb{E}\big[|\bar{X}_s|^{4m}\big]\big)\Delta^{1/2+\alpha}+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-1/2-\alpha}\mathbb{E}\Big[|\hat{X}_s-\bar{X}_s|^{2+4\alpha}\Big]\\
&+\frac{C\delta\Delta\big(\mathbb{E}\big[|\bar{X}_s|^{2+4\alpha+4m}\big]+1\big)}{\varepsilon\log\delta}\Bigg]ds.
\end{aligned}
\tag{42}
$$

Thanks to the condition $p_0\ge 2l\vee(2+4\alpha+4m)$, Theorem 2.4, Proposition 2.3, and Lemma 3.2, for any $T>0$, there exists a positive constant C_T such that for any $t\in[0,T]$, it holds that

Table 1. Four SDEs with their parameters.

Case	b	σ	p_0	L_1	γ	η	l	m	α
1	$-1+x-x^3$	$1+(1+x)x^{2/3}$	15	1	−1	18073	2	$\frac{4}{3}$	$\frac{1}{6}$
2	$-1+x-x^3$	$1+\sqrt{\frac{x^4+x^{4/3}}{14}}$	15	1	$\frac{13}{3}$	$\frac{5}{6}$	2	2	$\frac{1}{6}$
3	$-1-x-x^{7/3}$	$1+\sqrt{\frac{2x^2+x^{10/3}+x^{4/3}}{14}}$	15	−1	$\frac{13}{3}$	$\frac{5}{6}$	$\frac{4}{3}$	1	$\frac{1}{6}$
4	$-1-x-x^{7/3}$	$1+\sqrt{\frac{x^{10/3}+x^{4/3}}{14}}$	15	−1	$-\frac{1}{6}$	$\frac{11}{6}$	$\frac{4}{3}$	1	$\frac{1}{6}$

$$\mathbb{E}\left[e^{-L_1 t}|Y_t|\right] \le e^{-L_1 t}\varepsilon + C_T\left[\varepsilon + \Delta^{1/2} + \Delta + \frac{\varepsilon^{2\alpha}}{\log\delta} + \frac{\delta\Delta^{1/2+\alpha}}{\varepsilon\log\delta} + \frac{\delta\Delta}{\varepsilon\log\delta}\right]\int_0^t e^{-L_1 s}ds. \tag{43}$$

If $\alpha \in (0;\frac{1}{2}]$, choosing $\varepsilon = \Delta^{1/2}$, $\delta = 2$, we obtain

$$\sup_{0\le t\le T} \mathbb{E}[|Y_t|] \le C_T\Delta^{\alpha}.$$

If $\alpha = 0$, choosing $\varepsilon = \Delta^{1/4}, \delta = \Delta^{-1/4}$, we obtain

$$\sup_{0\le t\le T} \mathbb{E}[|Y_t|] \le \frac{C_T}{\log\frac{1}{\Delta}}.$$

We obtain (12). Note that if $L_1 < 0$ and $\gamma < 0$, we can choose the constant C_T in (43) such that it does not depend on T. Therefore, we also obtain (13). □

4. Examples

We consider four different SDEs with coefficients given in Table 1. These SDEs are chosen such that: $L_1 > 0, \gamma < 0$ in Case 1; $L_1 > 0, \gamma < 0$ in Case 2; $L_1 < 0, \gamma > 0$ in Case 3; and $L_1 < 0, \gamma < 0$ in Case 4. We choose $X_0 = 0$ in all the cases. It is straightforward to verify that these equations satisfy Assumptions **A1**–**A4** with constants $p_0, L_1, \gamma, \eta, l, m, \alpha$ given in Table 1 as well. In all these cases, $p_0 \ge 2l \vee (2 + 4\alpha + 4m)$, hence it follows from Theorem 2.7 that the tamed-adaptive Euler-Maruyama approximation scheme (2) converges in L^1-norm at the rate of order α in any finite time interval. Moreover, in Case 4, since $L_1 < 0$ and $\gamma < 0$, the tamed-adaptive Euler-Maruyama approximation scheme (2) converges in L^1-norm at the rate of order α in infinite time intervals.

In order to study the empirical rates of convergence of the tamed-adaptive Euler-Maruyama scheme, we consider

$$me(l) = \frac{1}{M}\sum_{k=1}^{M}|\hat{X}_1^{(l,k)} - \hat{X}_1^{(l+1,k)}|,$$

where for each $l \ge 2$, $(\hat{X}^{(l,k)})_{1\le k\le M}$ is a sequence of independent copies of $\hat{X}^{(l)}$ defined by Equations (2) and (3) with $\Delta = 2^{-l}$. Note that for each k and l, $\hat{X}_1^{(l,k)}$ and $\hat{X}_1^{(l+1,k)}$ must be generated on the same Brownian motion. This can be done by using the Algorithm 1 in [11].

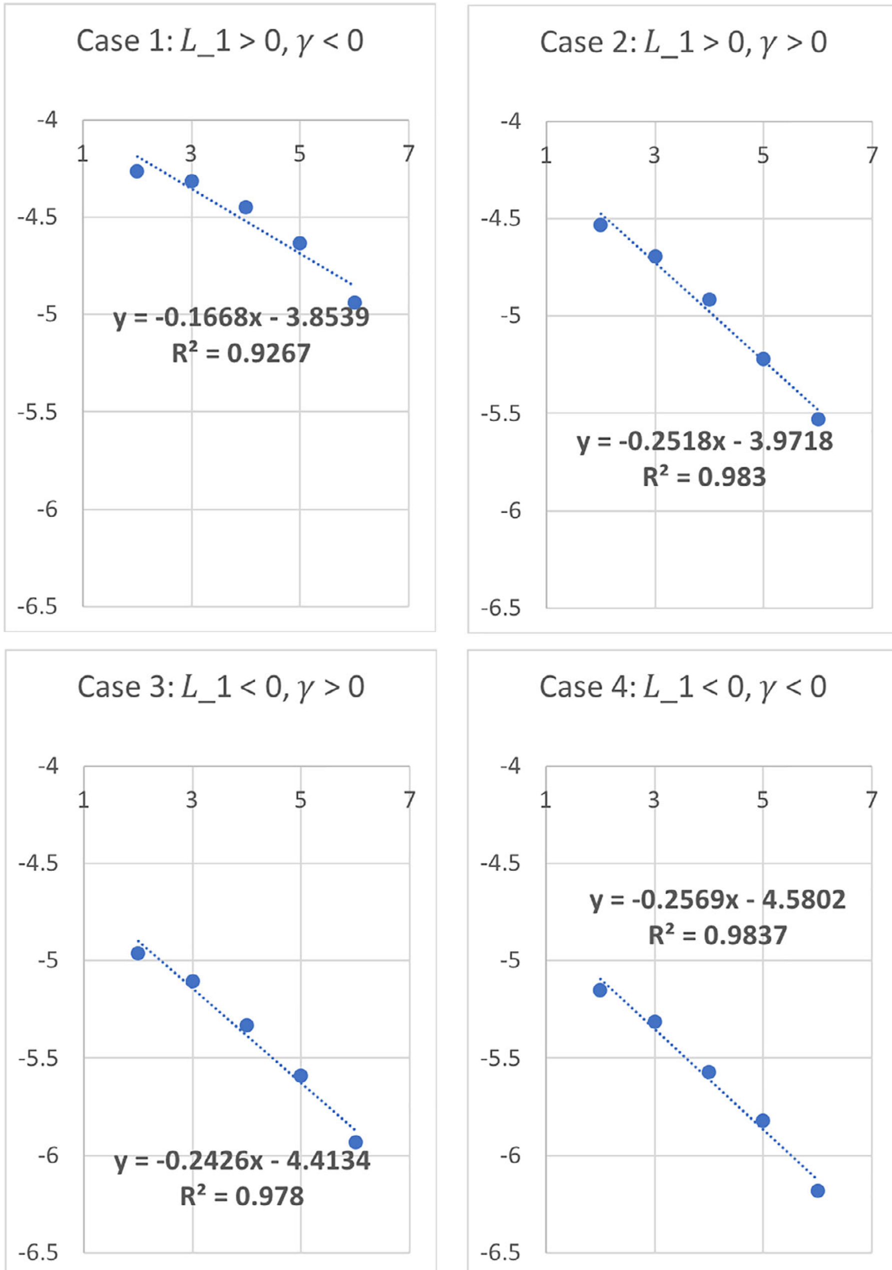

Figure 1. Values of $\log_2(me(l))$ for $l = 2, 3, 4, 5, 6$.

If $\hat{X}^{(l)}$ converges at the rate of order $\beta \in (0, +\infty)$ in L^1-norm, then there exists a constant $\beta > 0$ such that $2^{\beta l}\mathbb{E}[|X_1 - X_1^{(l)}|] = O(1)$, implying that $2^{\beta l}\mathbb{E}[|\hat{X}_1^{(l+1)} - \hat{X}_1^{(l)}|] = O(1)$ and vice-versa. In this case, we can write $\log_2 me(l) = -\beta l + C + o(1)$, for some constant $C \in \mathbb{R}$. Thus β can be estimated by the regression method.

Figure 1 shows the simulation result $\log_2 me(l)$ for $l = 2, \ldots, 6$. We draw the regression lines to estimate the empirical rates of convergence β in each case. In Case 1, the empirical rate of convergence, which is 0.1668, is almost the same as the theoretical rate, which is 1/6. In the other cases, the empirical rates are slightly better than the theoretical rate.

Note that in Case 4, the tamed-adaptive Euler-Maruyama approximation converges in infinitive time intervals while in other cases, it converges in any finite time intervals.

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Strong convergence in infinite time interval of tamed-adaptive Euler–Maruyama scheme for Lévy-driven SDEs with irregular coefficients

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Abstract
A tamed-adaptive Euler–Maruyama approximation scheme is proposed for Lévy-driven stochastic differential equations with locally Lipschitz continuous, polynomial growth drift, and locally Hölder continuous, polynomial growth diffusion coefficients. The new scheme converges in both finite and infinite time intervals under some suitable conditions on the regularity and the growth of the coefficients.

Keywords Euler–Maruyama approximation · Hölder continuous diffusion · Strong approximation · Polynomial growth coefficient

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1 Introduction

Throughout this paper, we consider the process $X = (X_t)_{t\geq 0}$ as a solution to the following stochastic differential equation (SDE) with jumps

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$$X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s + \int_0^t c\,(X_{s-})\,\mathrm{d}Z_s, \tag{1}$$

for $t \geq 0$, where $x_0 \in \mathbb{R}$, $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion and $Z = (Z_t)_{t\geq 0}$ is a one-dimensional centered pure jump Lévy process (independent of W) with Lévy measure ν satisfying $\int_{\mathbb{R}}(1 \wedge z^2)\nu(\mathrm{d}z) < +\infty$. Two processes W and Z are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W and Z and augmented by all the null sets in $\mathcal{F}$ so that it satisfies the usual conditions. The Lévy-Itô decomposition of Z takes the form

$$Z_t = \int_0^t \int_{\mathbb{R}_0} z(N(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s),$$

for any $t \geq 0$, where $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. Here, N is a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0))$ associated with the jumps of the Lévy process Z with intensity measure $\nu(\mathrm{d}z)\mathrm{d}t$. That is,

$$N(\mathrm{d}t, \mathrm{d}z) := \sum_{0\leq s\leq t} \mathbf{1}_{\{\Delta Z_s \neq 0\}} \delta_{(s,\Delta Z_s)}(\mathrm{d}s, \mathrm{d}z).$$

Here, the jump amplitude of Z at time s is defined as $\Delta Z_s := Z_s - Z_{s-} := Z_s - \lim_{u\uparrow s} Z_u$ for any $s > 0$, $\Delta Z_0 := 0$, $\delta_{(s,z)}$ denotes the Dirac measure at the point $(s, z) \in \mathbb{R}_+ \times \mathbb{R}_0$, and $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ denotes the Borel σ-algebra on $\mathbb{R}_+ \times \mathbb{R}_0$. Let $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$ denote the corresponding compensated Poisson random measure. The coefficients b, σ and c are real-valued measurable functions that will be specified later on. The integral equation of (1) can be written as

$$X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}_0} c\,(X_{s-})\,z\widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Such Lévy-driven SDEs arise in many applications (see Cont and Tankov 2003; Oksendal and Sulem 2007 and the references therein). Therefore, it is important to find effective methods to solve such SDEs numerically. The numerical approximation for Lévy-driven SDEs with Lipschitz continuous coefficients has been well studied (see Platen and Bruti-Liberati 2010; Jacod 2004). However, numerical analysis for Lévy-driven SDEs with non-Lipschitz coefficients is still a very active research area. It is well known that the classical Euler–Maruyama approximation may not converge when applying for SDEs with super-linearly growing coefficients (see Hutzenthaler et al. 2011). Several modified Euler–Maruyama schemes have been proposed for Lévy-driven SDEs with locally Lipschitz and super-linearly growing coefficients (see Higham and Kloeden 2005, 2006, 2007; Dareiotis et al. 2016; Kumar and Sabanis 2017a, b; Chen and Gan 2020; Chen et al. 2019; Deng et al. 2019; Li et al. 2021 and the references therein). The first explicit approximation for SDEs driven by Brownian motion with super-linearly growing drift coefficients is the tamed Euler–Maruyama schemes, which was introduced in Hutzenthaler et al. (2012) (see also Sabanis 2013; Hutzenthaler and Jentzen 2015, 2020). The strong convergence of Euler–Maruyama schemes for Lévy-driven SDEs with Hölder continuous diffusion coefficient has been studied in Li and Taguchi (2019a, b) and Yang and Wang (2017).

Note that, all the works mentioned above only considered the convergence of the approximation scheme in a finite time interval, say $[0, T]$ with $T < \infty$. For SDEs driven only by Brownian motions, the approximation in infinite time interval has just been studied recently by Fang and Giles (2020). They introduced an adaptive Euler–Maruyama approximation scheme and showed its strong convergence in the interval $[0, \infty)$ when applying for SDEs whose coefficients b and σ satisfy the contractive Lipschitz condition [Assumption 9 in Fang and Giles (2020)], b is locally Lipschitz and of polynomial growth, and σ is globally Lipschitz continuous. In Kieu et al. (2022), we introduced a tamed-adapted Euler–Maruyama approximation scheme and considered its strong convergence in L^1-norm on the time interval $[0, \infty)$ when applying for SDEs with locally Hölder continuous diffusion coefficients and superlinear growth coefficients.

This paper aims to propose a tamed-adaptive Euler–Maruyama approximation scheme for the Lévy-driven SDEs (1) where σ is locally Hölder continuous; σ and b are superlinear growth and c is Lipschitz continuous. We study the strong convergence of the scheme in both finite and infinite time intervals. Our finding extends the result in Fang and Giles (2020) and Kieu et al. (2022), which considered SDEs driven by Brownian motions, for a class of Lévy-driven SDEs. To the best of our knowledge, this is the first paper to construct a numerical scheme for Lévy-driven SDEs which converges in the infinite time interval $[0, \infty)$.

The rest of this paper is organized as follows. Section 2 provides a condition for the existence and uniqueness of the solution of Lévy-driven SDEs with irregular coefficients and an estimate for its moments. Section 3 introduces the tamed-adaptive Euler–Maruyama scheme and studies its convergence in both finite and infinite time intervals. Section 4 presents a numerical study for the tamed-adaptive scheme. The proof of the existence and uniqueness of the solution is given in "Appendix".

In all that follows, positive constants will be denoted by C whose value may change from one line to the next, and Q_n denotes polynomials of degree n.

2 Lévy-driven SDEs with irregular coefficients

The existence and uniqueness of the solution to Lévy-driven SDEs with non-Lipschitz coefficients have been studied by many authors (see Li and Mytnik 2011; Xi and Zhu 2019; Gou et al. 2020 and the references therein). In the following, we present another version of their result for SDEs with locally Hölder continuous diffusion coefficients and super-linearly growing drift and diffusion coefficients.

Theorem 2.1 *Assume that the coefficients b, c and σ satisfy the following conditions:*

C1. *There exists a positive constant L_0 such that*

$$|c(x)| \leq L_0(1 + |x|),$$

for any $x \in \mathbb{R}$.

C2. *For some $p_0 \in [2; +\infty)$, there exist constants $\gamma \in \mathbb{R}$, $\eta \in [0; +\infty)$ such that*

$$xb(x) + \frac{p_0 - 1}{2}\sigma^2(x) + \frac{c^2(x)}{2L_0}\int_{\mathbb{R}_0} |z| \left((1 + L_0|z|)^{p_0-1} - 1\right) \nu(\mathrm{d}z) \leq \gamma x^2 + \eta,$$

for any $x \in \mathbb{R}$.

C3. *Coefficient b is locally Lipschitz: for any $R > 0$, there exists a positive constant L_R such that*

$$|b(x) - b(y)| \leq L_R|x - y|,$$

for all $|x| \vee |y| \leq R$.

C4. *Coefficient σ is locally $\left(\alpha + \frac{1}{2}\right)$-Hölder continuous: for any $R > 0$, there exist positive constants L_R and $\alpha \in \left(0, \frac{1}{2}\right]$ such that*

$$|\sigma(x) - \sigma(y)| \leq L_R |x - y|^{1/2+\alpha},$$

for all $|x| \vee |y| \leq R$.

C5. *Coefficient c is locally Lipschitz: for any $R > 0$, there exists a positive constant L_R such that*

$$|c(x) - c(y)| \leq L_R |x - y|,$$

for all $|x| \vee |y| \leq R$.

Assume further that the Lévy measure satisfies $\int_{\mathbb{R}_0} |z|\nu(\mathrm{d}z) < \infty$ and $\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z) < \infty$. Then, the path-wise uniqueness holds for Eq. (1).

Moreover, suppose that there exist positive constants C and $\ell \in (0; \frac{p_0}{4}]$ such that

$$|b(x)| \vee |\sigma(x)| \vee |c(x)| \leq C \left(1 + |x|^{\ell}\right),$$

*for all $x \in \mathbb{R}$, where p_0 is defined in Condition **C2**. Then the Eq. (1) has a strong solution.*

The proof of Theorem 2.1 will be given in the "Appendix 5".

Remark 2.2 Since $(1 + L_0|z|)^x$ is an increasing function for $x \geq 1$, it follows from Condition **C2** that for any $p \in [2, p_0]$ and $x \in \mathbb{R}$,

$$xb(x) + \frac{p-1}{2}\sigma^2(x) + \frac{c^2(x)}{2L_0}\int_{\mathbb{R}_0} |z| \left((1 + L_0|z|)^{p-1} - 1\right) \nu(\mathrm{d}z) \leq \gamma x^2 + \eta.$$

We need some moment estimates of the exact solution.

Proposition 2.3 *Assume that coefficients b, c, σ and the Lévy measure ν satisfy conditions **C1, C2**, σ is bounded on every compact subset of $\mathbb{R}$, and*

C6. *$\int_{|z|>1} |z|^p\nu(\mathrm{d}z) < \infty$ for all $p \in [1; 2p_0]$ and $\int_{0<|z|\leq 1} |z|\nu(\mathrm{d}z) < \infty$.*

Assume further that $X = (X_t)_{t\geq 0}$ is a solution to Eq. (1). Then, for any $p \in (0, p_0]$, there exists a positive constant C_p such that for any $t \geq 0$,

$$\mathbb{E}\left[|X_t|^p\right] \leq \begin{cases} C_p(1 + e^{\gamma pt}) & \text{if } \gamma \neq 0, \\ C_p(1+t)^{p/2} & \text{if } \gamma = 0. \end{cases} \tag{2}$$

Note that when $\gamma < 0$, we have $\sup_{t\geq 0} \mathbb{E}\left[|X_t|^p\right] \leq 2C_p$.

We recall Kunita's inequality and Burkholder–Davis–Gundy's inequality with jumps.

Lemma 2.4 [Applebaum (2009, Theorem 4.4.23) and Zhu et al. (2019, Proposition 2.2)] *Let $\mathcal{P}$ be the progressive σ-algebra on $\mathbb{R}_+ \times \Omega$ and $\mathcal{B}(\mathbb{R}_0)$ be the Borel σ-algebra of $\mathbb{R}_0$. Assume that h is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0)$-measurable function such that $\int_0^T \int_{\mathbb{R}_0} |h(s,z)|^2\nu(\mathrm{d}z)\mathrm{d}s < \infty$ $\mathbb{P}$-a.s. for all $T \geq 0$. Then, for any $p \geq 2$, there exists a constant $C = C(p) > 0$ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \int_{\mathbb{R}_0} h(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right|^p\right]$$
$$\leq C\left(\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}_0} |h(s,z)|^2\nu(\mathrm{d}z)\mathrm{d}s\right)^{\frac{p}{2}}\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} |h(s,z)|^p\nu(\mathrm{d}z)\mathrm{d}s\right]\right).$$

Moreover, for any $1 \leq p < 2$, *there exists a constant* $C = C(p) > 0$ *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{R}_0} h(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right|^p\right] \leq C\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}_0}|h(s,z)|^2\nu(\mathrm{d}z)\mathrm{d}s\right)^{\frac{p}{2}}\right].$$

Proof of Proposition 2.3 Let $p \in [2, p_0]$ be an even natural number. Applying Itô's formula to $e^{-p\gamma t}X_t^p$, we have for any $t \geq 0$,

$$\begin{aligned}
e^{-p\gamma t}X_t^p &= x_0^p + p\int_0^t e^{-p\gamma s}\left(-\gamma X_s^p + X_s^{p-1}b\left(X_s\right) + \frac{p-1}{2}X_s^{p-2}\sigma^2\left(X_s\right)\right)\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-p\gamma s}\left[\left(X_s + c\left(X_s\right)z\right)^p - X_s^p - pX_s^{p-1}c\left(X_s\right)z\right]\nu(\mathrm{d}z)\mathrm{d}s \\
&\quad + p\int_0^t e^{-p\gamma s}X_s^{p-1}\sigma\left(X_s\right)\mathrm{d}W_s + \int_0^t\int_{\mathbb{R}_0} e^{-p\gamma s}\left[\left(X_{s-} + c\left(X_{s-}\right)z\right)^p - X_{s-}^p\right] \\
&\quad \widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned} \tag{3}$$

Then, applying (3) to $p = 2$, using **C2** and Remark 2.2, we get

$$\begin{aligned}
&e^{-2\gamma t}X_t^2 \\
&= x_0^2 + 2\int_0^t e^{-2\gamma s}\left[-\gamma X_s^2 + X_s b\left(X_s\right) + \frac{1}{2}\sigma^2\left(X_s\right) + \frac{1}{2}c^2\left(X_s\right)\int_{\mathbb{R}_0} z^2\nu\left(\mathrm{d}z\right)\right]\mathrm{d}s \\
&\quad + 2\int_0^t e^{-2\gamma s}X_s\sigma\left(X_s\right)\mathrm{d}W_s + \int_0^t\int_{\mathbb{R}_0} e^{-2\gamma s}\left[2X_{s-}c\left(X_{s-}\right)z + c^2\left(X_{s-}\right)z^2\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\
&\leq x_0^2 + 2\eta\int_0^t e^{-2\gamma s}\mathrm{d}s + 2\int_0^t e^{-2\gamma s}X_s\sigma\left(X_s\right)\mathrm{d}W_s \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-2\gamma s}\left[2X_{s-}c\left(X_{s-}\right)z + c^2\left(X_{s-}\right)z^2\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned} \tag{4}$$

Now, for each $N > 0$, we denote $\tau_N := \inf\{t \geq 0 : |X_t| \geq N\}$. Then, using (4), the fact that σ is bounded on every compact subset of $\mathbb{R}$ and Condition **C1**, we obtain

$$\mathbb{E}\left[e^{-2\gamma(t\wedge\tau_N)}X_{t\wedge\tau_N}^2\right] \leq x_0^2 + 2\eta\int_0^t e^{-2\gamma s}\mathrm{d}s. \tag{5}$$

This implies that

$$\mathbb{P}(\tau_N < t) \leq \left(x_0^2 + 2\eta\int_0^t e^{-2\gamma s}\mathrm{d}s\right)N^{-2},$$

which deduces that $\tau_N \uparrow \infty$ a.s. as $N \uparrow \infty$. Now, let $N \uparrow \infty$ and using Fatou's lemma for the left-hand side of (5), we obtain

$$\mathbb{E}\left[e^{-2\gamma t}X_t^2\right] \leq x_0^2 + 2\eta\int_0^t e^{-2\gamma s}\mathrm{d}s.$$

<u>If $\gamma = 0$:</u>

$$\mathbb{E}\left[X_t^2\right] \leq x_0^2 + 2\eta t.$$

<u>If $\gamma \neq 0$:</u>

$$\mathbb{E}\left[X_t^2\right] \leq \left(x_0^2 + \frac{\eta}{\gamma}\right)e^{2\gamma t} - \frac{\eta}{\gamma}.$$

Thus, (2) holds for $p = 2$. Thanks to Hölder's inequality, (2) is also valid for $p \in (0; 2]$.

Now, we suppose that (2) holds for all even integer $q \in [0, p-2]$ with even integer p, i.e.,

$$\mathbb{E}\left[X_t^q\right] \leq \begin{cases} C_q(1+e^{\gamma q t}) & \text{if } \gamma \neq 0, \\ C_q(1+t)^{q/2} & \text{if } \gamma = 0. \end{cases}$$

We shall show that (2) holds for p. For this, using the binomial theorem, we have

$$\begin{aligned} (X_s + c\left(X_s\right)z)^p - X_s^p - pX_s^{p-1}c\left(X_s\right)z =& \binom{p}{2} X_s^{p-2}c^2\left(X_s\right)z^2 \\ &+ \sum_{i=3}^{p} \binom{p}{i} X_s^{p-i}c^i\left(X_s\right)z^i. \end{aligned} \tag{6}$$

For all $3 \leq i \leq p$, using Condition **C1**, the binomial theorem and the fact that $|x|^{p-3} \leq \frac{1}{2}(|x|^{p-2} + |x|^{p-4})$ valid for any $x \in \mathbb{R}$, we get that

$$\begin{aligned} &X_s^{p-i}c^i\left(X_s\right) \\ &= X_s^{p-i}c^2\left(X_s\right)c^{i-2}\left(X_s\right) \\ &\leq |X_s|^{p-i}c^2\left(X_s\right)L_0^{i-2}\left(1+|X_s|\right)^{i-2} \\ &= |X_s|^{p-i}c^2\left(X_s\right)L_0^{i-2}\left(|X_s|^{i-2} + (i-2)|X_s|^{i-3} + \sum_{j=2}^{i-2}\binom{i-2}{j}|X_s|^{i-2-j}\right) \\ &= c^2\left(X_s\right)L_0^{i-2}\left(|X_s|^{p-2} + (i-2)|X_s|^{p-3} + \sum_{j=2}^{i-2}\binom{i-2}{j}|X_s|^{p-2-j}\right) \\ &\leq c^2\left(X_s\right)L_0^{i-2}\left(|X_s|^{p-2} + (i-2)\frac{1}{2}\left(|X_s|^{p-2} + |X_s|^{p-4}\right) + \sum_{j=2}^{i-2}\binom{i-2}{j}|X_s|^{p-2-j}\right) \\ &= c^2\left(X_s\right)L_0^{i-2}\left(\frac{i}{2}|X_s|^{p-2} + \frac{i-2}{2}|X_s|^{p-4} + \sum_{j=2}^{i-2}\binom{i-2}{j}|X_s|^{p-2-j}\right). \end{aligned}$$

This, combined with the fact that $\sum_{i=2}^{p}\binom{p}{i} i a^i = pa\left((1+a)^{p-1} - 1\right)$ valid for any $a \in \mathbb{R}$, we obtain that

$$\begin{aligned} &(X_s + c\left(X_s\right)z)^p - X_s^p - pX_s^{p-1}c\left(X_s\right)z \\ &\leq \binom{p}{2}|X_s|^{p-2}c^2\left(X_s\right)z^2 + c^2\left(X_s\right)|X_s|^{p-2}\sum_{i=3}^{p}\binom{p}{i}L_0^{i-2}\frac{i}{2}|z|^i \\ &\quad + c^2\left(X_s\right)\sum_{i=3}^{p}\binom{p}{i}L_0^{i-2}\left(\frac{i-2}{2}|X_s|^{p-4} + \sum_{j=2}^{i-2}\binom{i-2}{j}|X_s|^{p-2-j}\right)|z|^i \\ &= c^2\left(X_s\right)|X_s|^{p-2}\frac{1}{2L_0^2}\sum_{i=2}^{p}\binom{p}{i}i(L_0|z|)^i \end{aligned}$$

$$
\begin{aligned}
&+ c^2(X_s) \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \left(\frac{i-2}{2} |X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) |z|^i \\
&= c^2(X_s) |X_s|^{p-2} \frac{p}{2L_0} |z| \left((1+L_0|z|)^{p-1} - 1 \right) \\
&+ c^2(X_s) \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \left(\frac{i-2}{2} |X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) |z|^i. \qquad (7)
\end{aligned}
$$

Therefore, inserting (7) into (3), using Condition **C2**, Remark 2.2 and $c^2(x) \le 2L_0^2(1+x^2)$ for any $x \in \mathbb{R}$, we get

$$
\begin{aligned}
& e^{-p\gamma t} X_t^p \\
& \le x_0^p + p \int_0^t e^{-p\gamma s} X_s^{p-2} \\
& \left[-\gamma X_s^2 + X_s b(X_s) + \frac{p-1}{2} \sigma^2(X_s) + \frac{c^2(X_s)}{2L_0} \int_{\mathbb{R}_0} |z| \left((1+L_0|z|)^{p-1} - 1 \right) \nu(\mathrm{d}z) \right] \mathrm{d}s \\
& + \int_0^t e^{-p\gamma s} c^2(X_s) \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \left(\frac{i-2}{2} |X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) \int_{\mathbb{R}_0} |z|^i \nu(\mathrm{d}z) \mathrm{d}s \\
& + p \int_0^t e^{-p\gamma s} X_s^{p-1} \sigma(X_s) \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}_0} e^{-p\gamma s} \left[(X_{s-} + c(X_{s-}) z)^p - X_{s-}^p \right] \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
& \le x_0^p + p\eta \int_0^t e^{-p\gamma s} X_s^{p-2} \mathrm{d}s \\
& + 2 \int_0^t e^{-p\gamma s} \sum_{i=3}^{p} \binom{p}{i} L_0^{i} \left(\frac{i-2}{2} \left(|X_s|^{p-4} + |X_s|^{p-2} \right) + \sum_{j=2}^{i-2} \binom{i-2}{j} \left(|X_s|^{p-2-j} + |X_s|^{p-j} \right) \right) \\
& \times \int_{\mathbb{R}_0} |z|^i \nu(\mathrm{d}z) \mathrm{d}s + p \int_0^t e^{-p\gamma s} X_s^{p-1} \sigma(X_s) \mathrm{d}W_s \\
& + \int_0^t \int_{\mathbb{R}_0} e^{-p\gamma s} \left[(X_{s-} + c(X_{s-}) z)^p - X_{s-}^p \right] \widetilde{N}(\mathrm{d}s, \mathrm{d}z). \qquad (8)
\end{aligned}
$$

Now, it suffices to use the same argument as in the case $p = 2$ by replacing t by $t \wedge \tau_N$ in (8) and taking expectations on both sides to get that

$$
\begin{aligned}
\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)} X_{t\wedge\tau_N}^p \right] \le x_0^p + p\eta \int_0^t e^{-p\gamma s} \mathbb{E}\left[X_s^{p-2} \right] \mathrm{d}s \\
+ 2 \int_0^t e^{-p\gamma s} \sum_{i=3}^{p} \binom{p}{i} L_0^{i} \Bigg(\frac{i-2}{2} \left(\mathbb{E}\left[|X_s|^{p-4} \right] + \mathbb{E}\left[|X_s|^{p-2} \right] \right) \\
+ \sum_{j=2}^{i-2} \binom{i-2}{j} \left(\mathbb{E}\left[|X_s|^{p-2-j} \right] + \mathbb{E}\left[|X_s|^{p-j} \right] \right) \Bigg) \int_{\mathbb{R}_0} |z|^i \nu(\mathrm{d}z) \mathrm{d}s.
\end{aligned} \qquad (9)
$$

Then, using the fact that $\tau_N \uparrow \infty$ a.s. as $N \uparrow \infty$, letting $N \uparrow \infty$ on the left hand side of (9) and using Fatou's lemma and **C6**, we get

$$
\mathbb{E}\left[e^{-p\gamma t} X_t^p \right] \le x_0^p + p\eta \int_0^t e^{-p\gamma s} \mathbb{E}\left[X_s^{p-2} \right] \mathrm{d}s + C \int_0^t e^{-p\gamma s} \sum_{i=2}^{p} \mathbb{E}\left[|X_s|^{p-i} \right] \mathrm{d}s.
$$

Now, it suffices to use the inductive assumption.
If $\gamma = 0$:

$$\mathbb{E}\left[X_t^p\right] \leq x_0^p + C_p \int_0^t (1+s)^{p/2-1} \mathrm{d}s + C_p \sum_{i=2}^{p} \int_0^t (1+s)^{(p-i)/2} \mathrm{d}s$$
$$\leq C_p (1+t)^{p/2}.$$

If $\gamma \neq 0$:

$$\mathbb{E}\left[e^{-p\gamma t} X_t^p\right] \leq x_0^p + C_p \int_0^t e^{-p\gamma s} \left(1 + e^{\gamma(p-2)s}\right) \mathrm{d}s + C_p \sum_{i=2}^{p} \int_0^t e^{-p\gamma s} \left(1 + e^{\gamma(p-i)s}\right) \mathrm{d}s$$
$$\leq C_p + C_p e^{-p\gamma t}.$$

This implies that

$$\mathbb{E}\left[X_t^p\right] \leq C_p + C_p e^{\gamma p t}.$$

Therefore, (2) is valid for p. By the induction principle, (2) holds for any even natural number $p \in [2, p_0]$. Finally, using Hölder's inequality, we finish the proof for any $p \in (0, p_0]$. □

3 Tamed-adaptive Euler–Maruyama scheme

3.1 Definition of the tamed-adaptive Euler–Maruyama scheme

For each $\Delta \in (0, 1)$, the tamed-adaptive Euler–Maruyama discretisation of Eq. (1) is defined as follows

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0 = x_0, \quad t_{k+1} = t_k + h(\widehat{X}_{t_k})\Delta, \\ \widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k})(t_{k+1} - t_k) + \sigma_\Delta(\widehat{X}_{t_k})(W_{t_{k+1}} - W_{t_k}) + c_\Delta\left(\widehat{X}_{t_k}\right)\left(Z_{t_{k+1}} - Z_{t_k}\right), \end{cases} \tag{10}$$

where

$$h(x) = \frac{1}{(1 + |b(x)| + |\sigma(x)| + |x|^l)^2 + |c(x)|^{p_0}}, \tag{11}$$

Here, l, p_0 are some positive constants and $l \geq 1$, $p_0 \geq 2$. Moreover, c_Δ, σ_Δ are some approximations of c, σ, and their conditions will be specified later on.

The next result provides a sufficient condition for $t_k \to \infty$ as $k \to \infty$, which implies that the tamed adaptive approximation scheme (10) is well defined.

Proposition 3.1 *Suppose that there exist positive constants L and β such that the coefficients $b, c, \sigma, c_\Delta, \sigma_\Delta$ satisfy the following conditions*

T1. $|b(x)| \vee |\sigma(x)| \leq L\left(1 + |x|^\beta\right)$;
T2. $x(b(x) - b(0)) \leq L|x|^2$;
T3. $|\sigma_\Delta(x)| \leq L|\sigma(x)|$ *and* $|c_\Delta(x)| \leq |c(x)|$;
T4. $|\sigma_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$; $|c_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$ *and* $|b(x)c_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$;

for any $x \in \mathbb{R}$. *Assume further that the Lévy measure satisfies* $\int_{\mathbb{R}_0} z^2 \nu(\mathrm{d}z) < \infty$. *Then*

$$\lim_{k \to +\infty} t_k = +\infty \quad a.s.$$

We recall the approximation technique of Yamada and Watanabe (see Yamada and Watanabe 1971; Gyöngy and Rásonyi 2011). For each $\delta > 1$ and $\varepsilon > 0$, there exists a continuous function $\psi_{\delta\varepsilon} : \mathbb{R} \to \mathbb{R}_+$ with $\mathrm{supp}\psi_{\delta\varepsilon} \subset [\varepsilon/\delta; \varepsilon]$ such that

$$\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta\varepsilon}(z)\mathrm{d}z = 1; \quad 0 \le \psi_{\delta\varepsilon}(z) \le \frac{2}{z \log \delta}, \quad z > 0.$$

Then, we define

$$\phi_{\delta\varepsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta\varepsilon}(z)\mathrm{d}z\mathrm{d}y, \quad x \in \mathbb{R}.$$

It can be checked that $\phi_{\delta\epsilon}$ has the following useful properties: for any $x \in \mathbb{R}$,

YW1. $\phi'_{\delta\varepsilon}(x) = \frac{x}{|x|}\phi'_{\delta\varepsilon}(|x|)$,

YW2. $0 \le \left|\phi'_{\delta\varepsilon}(x)\right| \le 1$,

YW3. $|x| \le \varepsilon + \phi_{\delta\varepsilon}(x)$,

YW4. $\frac{\phi'_{\delta\varepsilon}(|x|)}{|x|} \le \frac{\delta}{\varepsilon}$,

YW5. $\phi''_{\delta\varepsilon}(|x|) = \psi_{\delta\varepsilon}(|x|) \le \frac{2}{|x|\log\delta} 1_{[\frac{\varepsilon}{\delta};\varepsilon]}(|x|) \le \frac{2\delta}{\varepsilon \log \delta}$.

Proof of Proposition 3.1 We will adapt the projection method in Kieu et al. (2022) (see also Fang and Giles 2020). For each $H > |x_0|$, a projected approximation scheme is defined as follows

$$\begin{cases} t_0^H = 0, \quad t_{k+1}^H = t_k^H + h(\widehat{X}^H_{t_k^H})\Delta, \\ \widehat{X}^H_{t_{k+1}^H} = P_H\left(\widehat{X}^H_{t_k^H} + b\left(\widehat{X}^H_{t_k^H}\right) h\left(\widehat{X}^H_{t_k^H}\right)\Delta \right. \\ \qquad \left. +\sigma_\Delta\left(\widehat{X}^H_{t_k^H}\right)(W_{t_{k+1}^H} - W_{t_k^H}) + c_\Delta\left(\widehat{X}^H_{t_k^H}\right)(Z_{t_{k+1}^H} - Z_{t_k^H})\right), \end{cases}$$

where $P_H(Y) := \min\left(1, \frac{H}{|Y|}\right)Y$. Observe that $\left|\widehat{X}^H_{t_k^H}\right| \le H$ for all k. Thus $h(\widehat{X}^H_{t_k^H})\Delta \ge C(H, L, l, m)\Delta$, which implies that $t_k^H \uparrow \infty$ as $k \to \infty$. We note that for each k, t_k^H is a stopping time and t_{k+1}^H is $\mathcal{F}_{t_k^H}$-measurable. Set $\underline{t}^H = \max\{t_k^H : t_k^H \le t\}$. Then $\underline{t}^H$ is also a stopping time. Throughout the proof, we use the following result which is a consequence of the strong Markov property of W

$$\mathbb{E}\left[\left(W_t - W_{\underline{t}^H}\right)^r |\mathcal{F}_{\underline{t}^H}\right] = \begin{cases} 0 & \text{if } r \text{ is an odd integer} \\ \alpha_r(t - \underline{t}^H)^{r/2} & \text{if } r \text{ is an even integer,} \end{cases} \tag{12}$$

for some positive constant α_r.

The continuous approximation is given by

$$\widehat{X}^H_t = P_H\left(\widehat{X}^H_{\underline{t}^H} + b\left(\widehat{X}^H_{\underline{t}^H}\right)\left(t - \underline{t}^H\right) + \sigma_\Delta\left(X^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right) + c_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)\right).$$

First, the fact that $P_H(Y) = \min(1, H/|Y|)Y$ implies $\phi_{\delta\varepsilon}(P_H(Y)) \le \phi_{\delta\varepsilon}(Y)$. Therefore,

$$\phi_{\delta\varepsilon}\left(\widehat{X}_t^H\right)=\phi_{\delta\varepsilon}\Big[P_H\left(\widehat{X}_{\underline{t}^H}^H+b\left(\widehat{X}_{\underline{t}^H}^H\right)(t-\underline{t}^H)+\sigma_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(W_t-W_{\underline{t}^H}\right)+c_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(Z_t-Z_{\underline{t}^H}\right)\right)\Big]$$
$$\leq \phi_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H+b\left(\widehat{X}_{\underline{t}^H}^H\right)(t-\underline{t}^H)+\sigma_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(W_t-W_{\underline{t}^H}\right)+c_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(Z_t-Z_{\underline{t}^H}\right)\right).$$

Next, using Taylor's expansion, there exists an $(\mathcal{F}_t)$-adapted process (ξ_t) such that

$$\begin{aligned}
&\phi_{\delta\varepsilon}\left(\widehat{X}_t^H\right)\\
&\leq \phi_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)+\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)\Big[b\left(\widehat{X}_{\underline{t}^H}^H\right)(t-\underline{t}^H)\\
&\quad+\sigma_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(W_t-W_{\underline{t}^H}\right)+c_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(Z_t-Z_{\underline{t}^H}\right)\Big]+\frac{1}{2}\phi''_{\delta\varepsilon}(\xi_t)\Big[b\left(\widehat{X}_{\underline{t}^H}^H\right)(t-\underline{t}^H)\\
&\quad+\sigma_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(W_t-W_{\underline{t}^H}\right)+c_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(Z_t-Z_{\underline{t}^H}\right)\Big]^2.
\end{aligned}$$

Recall that $|\phi''_{\delta\varepsilon}(x)|\leq\frac{2\delta}{\varepsilon\log\delta}$. Thus,

$$\begin{aligned}
&\phi_{\delta\varepsilon}\left(\widehat{X}_t^H\right)\\
&\leq \phi_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)+\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)\Big[b\left(\widehat{X}_{\underline{t}^H}^H\right)-b(0)\Big](t-\underline{t}^H)+\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)b\,(0)\,(t-\underline{t}^H)\\
&\quad+\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)\sigma_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(W_t-W_{\underline{t}^H}\right)+\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)c_\Delta\left(\widehat{X}_{\underline{t}^H}^H\right)\left(Z_t-Z_{\underline{t}^H}\right)\\
&\quad+\frac{3\delta}{\varepsilon\log\delta}\left(b^2\left(\widehat{X}_{\underline{t}^H}^H\right)(t-\underline{t}^H)^2+\sigma_\Delta^2\left(\widehat{X}_{\underline{t}^H}^H\right)\left(W_t-W_{\underline{t}^H}\right)^2\right.\\
&\quad\left.+c_\Delta^2\left(\widehat{X}_{\underline{t}^H}^H\right)\left(Z_t-Z_{\underline{t}^H}\right)^2\right).
\end{aligned}$$

Using (11), we get

$$b^2(\widehat{X}_{\underline{t}^H}^H)(t-\underline{t}^H)\leq b^2\left(\widehat{X}_{\underline{t}^H}^H\right)h\left(\widehat{X}_{\underline{t}^H}^H\right)\Delta\leq\Delta.$$

Using **YW3**, **T2** and the fact that $|x|\phi'_{\delta\varepsilon}(|x|)\leq\phi_{\delta\varepsilon}(x)+\varepsilon$, we get

$$\begin{aligned}
&\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)\Big[b\left(\widehat{X}_{\underline{t}^H}^H\right)-b(0)\Big](t-\underline{t}^H)\\
&=\frac{\phi'_{\delta\varepsilon}\left(|\widehat{X}_{\underline{t}^H}^H|\right)}{|\widehat{X}_{\underline{t}^H}^H|}\widehat{X}_{\underline{t}^H}^H\Big[b\left(\widehat{X}_{\underline{t}^H}^H\right)-b(0)\Big](t-\underline{t}^H)\\
&\leq L\left|\widehat{X}_{\underline{t}^H}^H\right|\phi'_{\delta\varepsilon}\left(|\widehat{X}_{\underline{t}^H}^H|\right)(t-\underline{t}^H)\\
&\leq L\left(\phi_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)+\varepsilon\right)(t-\underline{t}^H).
\end{aligned}$$

Now, using **T4** and the fact that $\mathrm{d}W_t^2=2W_t\mathrm{d}W_t+\mathrm{d}t$, we have

$$\sigma_\Delta^2\left(\widehat{X}_{\underline{t}^H}^H\right)\left(W_t-W_{\underline{t}^H}\right)^2\leq\frac{L^2}{\Delta}\left(2\int_{\underline{t}^H}^t\left(W_s-W_{\underline{s}^H}\right)\mathrm{d}W_s+(t-\underline{t}^H)\right).$$

Moreover, the Itô's formula yields

$$\left(Z_t-Z_{\underline{t}^H}\right)^2=(t-\underline{t}^H)\int_{\mathbb{R}_0}z^2\nu(\mathrm{d}z)+\int_{\underline{t}^H}^t\int_{\mathbb{R}_0}\Big[z^2+2z\left(Z_{s-}-Z_{\underline{t}^H}\right)\Big]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).$$

This, together with **T4**, gives

$$
\begin{aligned}
& c_\Delta^2\left(\widehat{X}_{\underline{t}^H}^H\right)\left(Z_t-Z_{\underline{t}^H}\right)^2 \\
& \quad \leq \frac{L^2\left(t-\underline{t}^H\right)}{\Delta} \int_{\mathbb{R}_0} z^2 \nu(\mathrm{d} z)+\frac{L^2}{\Delta} \int_{\underline{t}^H}^t \int_{\mathbb{R}_0}\left[z^2+2 z\left(Z_{s-}-Z_{\underline{s}^H}\right)\right] \widetilde{N}(\mathrm{d} s, \mathrm{d} z).
\end{aligned}
$$

Consequently, we have shown that

$$
\begin{aligned}
& \phi_{\delta\varepsilon}\left(\widehat{X}_t^H\right) \\
& \quad \leq\left[1+L\left(t-\underline{t}^H\right)\right] \phi_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right) \\
& \quad +\left(L\varepsilon+|b(0)|+\frac{3\delta\Delta}{\varepsilon\log\delta}+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\int_{\mathbb{R}_0} z^2\nu(dz)\right)\left(t-\underline{t}^H\right) \\
& \quad +\int_{\underline{t}^H}^t\left[\phi_{\delta e}^{\prime}\left(\widehat{X}_{\underline{s}^H}^H\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)+\frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right]\mathrm{d}W_s \\
& \quad +\int_{\underline{t}^H}^t\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}^{\prime}\left(\widehat{X}_{\underline{s}^H}^H\right)c_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)z+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2+2z\left(Z_{s-}-Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
\tag{13}
$$

Note that $e^{-Lt}(1+L(t-\underline{t}^H)) \leq e^{-L\underline{t}^H}$. Then, multiplying by e^{-Lt} in both sides of (13), we have

$$
\begin{aligned}
& e^{-Lt}\phi_{\delta\varepsilon}\left(\widehat{X}_t^H\right) \\
& \quad \leq e^{-L\underline{t}^H}\phi_{\delta\varepsilon}\left(\widehat{X}_{\underline{t}^H}^H\right)+C(b(0),L,\Delta,\varepsilon,\delta)\int_{\underline{t}^H}^t e^{-Ls}\mathrm{d}s \\
& \quad +e^{-Lt}\int_{\underline{t}^H}^t\left[\phi_{\delta e}^{\prime}\left(\widehat{X}_{\underline{s}^H}^H\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)+\frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right]\mathrm{d}W_s \\
& \quad +e^{-Lt}\int_{\underline{t}^H}^t\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}^{\prime}\left(\widehat{X}_{\underline{s}^H}^H\right)c_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)z\right. \\
& \quad \left.+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2+2z\left(Z_{s-}-Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
\tag{14}
$$

Now, from (14), we get

$$
\begin{aligned}
& e^{-Lt_{k+1}^H}\phi_{\delta\varepsilon}\left(\widehat{X}_{t_{k+1}^H}^H\right) \\
& \quad \leq e^{-Lt_k^H}\phi_{\delta\varepsilon}\left(\widehat{X}_{t_k^H}^H\right)+C(b(0),L,\Delta,\varepsilon,\delta)\int_{t_k^H}^{t_{k+1}^H} e^{-Ls}\mathrm{d}s \\
& \quad +e^{-Lt_{k+1}^H}\int_{t_k^H}^{t_{k+1}^H}\left[\phi_{\delta e}^{\prime}\left(\widehat{X}_{\underline{s}^H}^H\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)+\frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right]\mathrm{d}W_s \\
& \quad +e^{-Lt_{k+1}^H}\int_{t_k^H}^{t_{k+1}^H}\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}^{\prime}\left(\widehat{X}_{\underline{s}^H}^H\right)c_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)z\right. \\
& \quad \left.+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2+2z\left(Z_{s-}-Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
\tag{15}
$$

Summing (15) over multiple timesteps and adding (14), we obtain

$$
\begin{aligned}
e^{-Lt}\phi_{\delta\varepsilon}\left(\widehat{X}_t^H\right) &\leq \phi_{\delta\varepsilon}(x_0) + C(b(0), L, \Delta, \varepsilon, \delta)\int_0^t e^{-Ls}\mathrm{d}s \\
&+ \int_0^t e^{-L\left(\underline{s}^H + h\left(\widehat{X}_{\underline{s}^H}^H\right)\right)}\left[\phi_{\delta e}'\left(\widehat{X}_{\underline{s}H}^H\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right) + \frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s - W_{\underline{s}^H}\right)\right]\mathrm{d}W_s \\
&+ \int_0^t\int_{\mathbb{R}_0} e^{-L\left(\underline{s}^H + h\left(\widehat{X}_{\underline{s}^H}^H\right)\right)}\left[\phi_{\delta\varepsilon}'\left(\widehat{X}_{\underline{s}^H}^H\right)c_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)z\right. \\
&\left.+ \frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2 + 2z\left(Z_{s-} - Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \left(e^{-Lt} - e^{-L\left(\underline{t}^H + h\left(\widehat{X}_{\underline{t}^H}^H\right)\right)}\right) \\
&\times \int_{\underline{t}^H}^t\left[\phi_{\delta\varepsilon}'\left(\widehat{X}_{\underline{s}^H}^H\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right) + \frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s - W_{\underline{s}^H}\right)\right]\mathrm{d}W_s \\
&+ \left(e^{-Lt} - e^{-L\left(\underline{t}^H + h\left(\widehat{X}_{\underline{t}^H}^H\right)\right)}\right)\int_{\underline{t}^H}^t\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}'\left(\widehat{X}_{\underline{s}^H}^H\right)c_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)z\right. \\
&\left.+ \frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2 + 2z\left(Z_{s-} - Z_{\underline{s}}^H\right)\right)\right]\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\leq \phi_{\delta\varepsilon}(x_0) + C(b(0), L, \Delta, \varepsilon, \delta)\int_0^t e^{-Ls}\mathrm{d}s \\
&+ \int_0^t e^{-L\left(\underline{s}^H + h\left(\widehat{X}_{\underline{s}^H}^H\right)\right)}\left[\phi_{\delta e}'\left(\widehat{X}_{\underline{s}H}^H\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right) + \frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s - W_{\underline{s}^H}\right)\right]\mathrm{d}W_s \\
&+ \int_0^t\int_{\mathbb{R}_0} e^{-L\left(\underline{s}^H + h\left(\widehat{X}_{\underline{s}^H}^H\right)\right)}\left[\phi_{\delta\varepsilon}'\left(\widehat{X}_{\underline{s}^H}^H\right)c_\Delta\left(\widehat{X}_{\underline{s}^H}^H\right)z\right. \\
&\left.+ \frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2 + 2z\left(Z_{s-} - Z_{\underline{s}^H}\right)\right)\right] \\
&\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&+ L\Delta\left(\frac{2L}{\sqrt{\Delta}}\sup_{0\leq s\leq t}|W_s| + \frac{12\delta L^2}{\Delta\varepsilon\log\delta}\sup_{0\leq s\leq t}|W_s|^2\right) \\
&+ L\Delta\left(\frac{2L}{\sqrt{\Delta}}\sup_{0\leq s\leq t}|Z_s| + \frac{12\delta L^2}{\Delta\varepsilon\log\delta}\sup_{0\leq s\leq t}|Z_s|^2\right).
\end{aligned}
$$

Now, using Lemma 2.4, we have

$$
\mathbb{E}\left[\sup_{0\leq s\leq t}|Z_s|\right] \leq C\sqrt{t}\left(\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z)\right)^{\frac{1}{2}}; \quad \mathbb{E}\left[\sup_{0\leq s\leq t}Z_s^2\right] \leq Ct\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z).
$$

Hence, for any the stopping time $\tau \leq t$, it follows from the modulus of continuity of W and the Condition **C6** that there exists a positive constant $C = C(x_0, \varepsilon, \delta, \Delta, L, b(0), t)$, which does not depend on H such that

$$
\mathbb{E}\left[e^{-L\tau}\left|\widehat{X}_\tau^H\right|\right] \leq C(x_0, \varepsilon, \delta, \Delta, L, b(0), t).
$$

From this point forward, by repeating the argument in the proof of Proposition 2.1 in Kieu et al. (2022) we obtain the desired result. □

Under the assumptions of Proposition 3.1, the nearest time point before t is defined by $\underline{t} := \max\{t_n : t_n \le t\}$, and the number of time steps approximation up to time t is defined by $N_t := \max\{n : t_n \le t\}$. Observe that $\underline{t}$ is a stopping time. Then, the standard continuous interpolant is defined by

$$\widehat{X}_t = \widehat{X}_{\underline{t}} + b\left(\widehat{X}_{\underline{t}}\right)(t-\underline{t}) + \sigma_\Delta\left(\widehat{X}_{\underline{t}}\right)\left(W_t - W_{\underline{t}}\right) + c_\Delta\left(\widehat{X}_{\underline{t}-}\right)\left(Z_t - Z_{\underline{t}}\right). \tag{16}$$

Hence, $\widehat{X} = (\widehat{X}_t)_{t\ge 0}$ is the solution of the following SDE

$$\mathrm{d}\widehat{X}_t = b(\widehat{X}_{\underline{t}})\mathrm{d}t + \sigma_\Delta\left(\widehat{X}_{\underline{t}}\right)\mathrm{d}W_t + c_\Delta\left(\widehat{X}_{\underline{t}-}\right)\mathrm{d}Z_t, \qquad \widehat{X}_0 = x_0. \tag{17}$$

3.2 Moments of the tamed-adaptive Euler–Maruyama scheme

We first provide the following preliminary estimate on the moments of $\widehat{X}$.

Lemma 3.2 *Assume that Conditions* ***T1–T4*** *and* ***C6*** *hold. Then for any $p \in [1; 2p_0]$ and $T > 0$, there exists a positive constant $C(p, L, T, x_0, \Delta)$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |\widehat{X}_t|^p\right] \le C(p, L, T, x_0, \Delta).$$

Proof Using the property **YW3** and applying Itô's formula for $e^{-Lt}\phi_{\delta\varepsilon}(\widehat{X}_t)$, we get

$$\begin{aligned}
e^{-Lt}|\widehat{X}_t| &\le e^{-Lt}\varepsilon + e^{-Lt}\phi_{\delta\varepsilon}(\widehat{X}_t)\\
&\le \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\left[-L\phi_{\delta\varepsilon}(\widehat{X}_s) + \phi'_{\delta\varepsilon}(\widehat{X}_s)b(\widehat{X}_{\underline{s}}) + \frac{1}{2}\phi''_{\delta\varepsilon}(\widehat{X}_s)\sigma^2_\Delta(\widehat{X}_{\underline{s}})\right]\mathrm{d}s\\
&\quad + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\widehat{X}_s)\sigma_\Delta(\widehat{X}_{\underline{s}})\mathrm{d}W_s\\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_s + c_\Delta(\widehat{X}_{\underline{s}})z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_s\right) - \phi'_{\delta\varepsilon}\left(\widehat{X}_s\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right]\nu(\mathrm{d}z)\mathrm{d}s\\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_{s-} + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_{s-}\right)\right]\widetilde{N}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Now, applying Taylor's expansion for $\phi'_{\delta\varepsilon}$ and using **T2**, **YW2**, **YW5** and Eq. (16), there exists an $(\mathcal{F}_s)$-adapted process $\xi = (\xi_s)$ such that

$$\begin{aligned}
\phi'_{\delta\varepsilon}(\widehat{X}_s)b(\widehat{X}_{\underline{s}}) &= \left(\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right) + \phi''_{\delta\varepsilon}(\xi_s)\left(\widehat{X}_s - \widehat{X}_{\underline{s}}\right)\right)b\left(\widehat{X}_{\underline{s}}\right)\\
&= \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)b\left(\widehat{X}_{\underline{s}}\right) + \phi''_{\delta\varepsilon}(\xi_s)\left(b\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s})\right.\\
&\quad \left.+\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(W_s - W_{\underline{s}}\right) + c_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(Z_s - Z_{\underline{s}}\right)\right)b\left(\widehat{X}_{\underline{s}}\right)\\
&= \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)\left(b\left(\widehat{X}_{\underline{s}}\right) - b(0)\right) + \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)b(0) + \phi''_{\delta\varepsilon}(\xi_s)\left(b^2\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s})\right.\\
&\quad \left.+b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(W_s - W_{\underline{s}}\right) + b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(Z_s - Z_{\underline{s}}\right)\right)\\
&= \frac{\phi'_{\delta\varepsilon}\left(\left|\widehat{X}_{\underline{s}}\right|\right)}{\left|\widehat{X}_{\underline{s}}\right|}\widehat{X}_{\underline{s}}\left(b\left(\widehat{X}_{\underline{s}}\right) - b(0)\right) + \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)b(0) + \phi''_{\delta\varepsilon}(\xi_s)\left(b^2\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s})\right.\\
&\quad \left.+b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(W_s - W_{\underline{s}}\right) + b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(Z_s - Z_{\underline{s}}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq L\left|\widehat{X}_{\underline{s}}\right| + |b(0)| + \frac{C\delta\Delta}{\varepsilon\log\delta} + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right| \\
&\quad + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right|.
\end{aligned}
$$

Again, applying Taylor's expansion for $\phi_{\delta\varepsilon}$ and using **YW5**, **T4**, there exists an $(\mathcal{F}_s)$-adapted process $\theta = (\theta_s)$ such that

$$
\begin{aligned}
&\phi_{\delta\varepsilon}\left(\widehat{X}_s + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_s\right) - \phi'_{\delta\varepsilon}\left(\widehat{X}_s\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)z \\
&\quad = \frac{1}{2}\phi''_{\delta\varepsilon}(\theta_s)\,c_\Delta^2\left(\widehat{X}_{\underline{s}}\right)z^2 \leq \frac{\delta}{\varepsilon\log\delta}\frac{L^2}{\Delta}z^2.
\end{aligned}
$$

Hence, we have shown that

$$
\begin{aligned}
e^{-Lt}|\widehat{X}_t| &\leq \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\Bigg[-L\phi_{\delta\varepsilon}(\widehat{X}_s) + L\left|\widehat{X}_{\underline{s}}\right| + |b(0)| + \frac{C\delta\Delta}{\varepsilon\log\delta} \\
&\quad + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right| \\
&\quad + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right| + \frac{L\delta}{\Delta\varepsilon\log\delta}\Bigg]\mathrm{d}s \\
&\quad + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\widehat{X}_s)\sigma_\Delta(\widehat{X}_{\underline{s}})\mathrm{d}W_s + \frac{L^2\delta}{\Delta\varepsilon\log\delta}\int_0^t e^{-Ls}\mathrm{d}s\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z) \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_{s-} + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_{s-}\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
$$

Next, using **YW3** and (16), we have

$$
\begin{aligned}
&-\phi_{\delta\varepsilon}(\widehat{X}_s) + \left|\widehat{X}_{\underline{s}}\right| \leq -|\widehat{X}_s| + \varepsilon + \left|\widehat{X}_{\underline{s}}\right| \\
&\quad \leq \varepsilon + |b(\widehat{X}_{\underline{s}})(s - \underline{s})| + |\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})| + |c_\Delta(\widehat{X}_{\underline{s}})(Z_s - Z_{\underline{s}})|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
e^{-Lt}|\widehat{X}_t| &\leq (1 + Lt)\varepsilon + \phi_{\delta\varepsilon}(x_0) \\
&\quad + \int_0^t Le^{-Ls}\left(|b(\widehat{X}_{\underline{s}})(s - \underline{s})| + |\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})| + |c_\Delta(\widehat{X}_{\underline{s}})(Z_s - Z_{\underline{s}})|\right)\mathrm{d}s \\
&\quad + \int_0^t e^{-Ls}\Bigg[|b(0)| + \frac{C\delta\Delta}{\varepsilon\log\delta} + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right| \\
&\quad + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right| + \frac{C\delta}{\Delta\varepsilon\log\delta}\Bigg]\mathrm{d}s \\
&\quad + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\widehat{X}_s)\sigma_\Delta(\widehat{X}_{\underline{s}})\mathrm{d}W_s + \frac{CL^2\delta}{\Delta\varepsilon\log\delta}\int_0^t e^{-Ls}\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_{s-} + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_{s-}\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z). \qquad (18)
\end{aligned}
$$

Using (12), for any $p \in [1; 2p_0]$, there exists a constant $C(p) > 0$ such that

$$
\begin{aligned}
&\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right|^p\right] \\
&\quad = \mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\mathbb{E}\left[|W_s - W_{\underline{s}}|^p\middle|\mathcal{F}_{\underline{s}}\right]\right]
\end{aligned}
$$

$$\leq C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p (s-\underline{s})^{p/2}\right]$$
$$\leq C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p \left|h(\widehat{X}_{\underline{s}})\right|^{p/2}\Delta^{p/2}\right].$$

Next, using the Burkholder–Davis–Gundy's inequality with jumps and **C6**, for any $p \in [1; 2p_0]$, there exists a constant $C(p) > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right|^p\right]\\
&= \mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p \mathbb{E}\left[|Z_s - Z_{\underline{s}}|^p\Big|\mathcal{F}_{\underline{s}}\right]\right]\\
&\leq C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\left(\int_{\underline{s}}^s\int_{\mathbb{R}_0}|z|^{2\vee p}\nu(\mathrm{d}z)\mathrm{d}s\right)^{1\wedge p/2}\right]\\
&= C(p)\left(\int_{\mathbb{R}_0}|z|^{2\vee p}\nu(\mathrm{d}z)\right)^{1\wedge p/2}\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p(s-\underline{s})^{1\wedge p/2}\right]\\
&\leq C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\left|h(\widehat{X}_{\underline{s}})\right|^{1\wedge p/2}\Delta^{1\wedge p/2}\right].
\end{aligned}$$

Thanks to (11) and Conditions **T3, T4**, we have

$$\begin{aligned}
&\max\left\{\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right|^p\right];\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right|^p\right]\right\}\\
&\leq C(p, L, \Delta).
\end{aligned}$$

Therefore, by choosing $\varepsilon = 1$, $\delta = 2$ in (18), it follows from **T1–T4**, Hölder's inequality and Burkholder–Davis–Gundy's inequality that for any $T > 0$, there exists a positive constant $C(p, L, T, x_0, \Delta) < \infty$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\widehat{X}_t|^p\right] \leq C(p, L, T, x_0, \Delta).$$

This finishes the proof. □

The following estimates show how moments of $\hat{X}_t$ depend on t.

Theorem 3.3 *Assume that Conditions* ***T1–T4*** *and* ***C6*** *hold, and for some $p_0 \in [2, +\infty)$, there exist constants $\gamma \in \mathbb{R}$, $\eta \in [0, +\infty)$ such that for all $x \in \mathbb{R}$,*

$$xb(x) + \frac{p_0 - 1}{2}\sigma_\Delta^2(x) + \frac{c_\Delta^2(x)}{2L_0}\int_{\mathbb{R}_0}|z|\left((1 + L_0|z|)^{p_0-1} - 1\right)\nu(\mathrm{d}z) \leq \gamma x^2 + \eta. \quad (19)$$

Then, for any positive integer $k \leq p_0/2$, there exists a positive constant $C = C(x_0, k, \eta, \gamma, L, L_0, p_0)$ which does not depend neither on t nor on Δ such that

$$\mathbb{E}\left[|\widehat{X}_t|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}|^{2k}\right] \leq \begin{cases} Ce^{2k\gamma t} & \text{if } \gamma > 0,\\ C(1+t)^k & \text{if } \gamma = 0,\\ C & \text{if } \gamma < 0.\end{cases} \quad (20)$$

Remark 3.4 Since $(1+L_0|z|)^x$ is an increasing function for $x \geq 1$, we deduce from condition (19) that for any $p \in [2, p_0]$ and $x \in \mathbb{R}$,

$$xb(x) + \frac{p - 1}{2}\sigma_\Delta^2(x) + \frac{c_\Delta^2(x)}{2L_0}\int_{\mathbb{R}_0}|z|\left((1 + L_0|z|)^{p-1} - 1\right)\nu(\mathrm{d}z) \leq \gamma x^2 + \eta.$$

Proof of Theorem 3.3 Using Hölder's inequality, it is sufficient to show (20) for a positive integer k and $k \leq p_0/2$. We will use the induction method.

Firstly, for $k=1$, applying Itô's formula to $e^{-2\gamma t}\widehat{X}_t^2$, we get that

$$
\begin{aligned}
e^{-2\gamma t}\widehat{X}_t^2 = x_0^2 &+ 2\int_0^t e^{-2\gamma s}\left(-\gamma \widehat{X}_s^2 + \widehat{X}_s b(\widehat{X}_{\underline{s}}) + \frac{1}{2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) + \frac{1}{2}c_\Delta^2(\widehat{X}_{\underline{s}})\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z)\right)\mathrm{d}s \\
&+ 2\int_0^t e^{-2\gamma s}\widehat{X}_s \sigma_\Delta(\widehat{X}_{\underline{s}})\mathrm{d}W_s \\
&+ \int_0^t\int_{\mathbb{R}_0} e^{-2\gamma s}\left[2\widehat{X}_{\underline{s}-}c_\Delta\left(\widehat{X}_{\underline{s}-}\right)z + c_\Delta^2\left(\widehat{X}_{\underline{s}-}\right)z^2\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z). \qquad (21)
\end{aligned}
$$

It follows from (16) that

$$
\begin{aligned}
\widehat{X}_s^2 = \widehat{X}_{\underline{s}}^2 &+ 2\widehat{X}_{\underline{s}}\left(b\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s}) + \sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(W_s - W_{\underline{s}}\right) + c_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(Z_s - Z_{\underline{s}}\right)\right) \\
&+ \left(b\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s}) + \sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(W_s - W_{\underline{s}}\right) + c_\Delta\left(\widehat{X}_{\underline{s}}\right)\left(Z_s - Z_{\underline{s}}\right)\right)^2.
\end{aligned}
$$

Using **T3**, **T4**, **C6**, (12), and (11),

$$
\begin{aligned}
\max\Big\{&|\widehat{X}_{\underline{s}} b(\widehat{X}_{\underline{s}})(s-\underline{s})|;\ b^2(\widehat{X}_{\underline{s}})(s-\underline{s})^2;\ \mathbb{E}\left[\sigma_\Delta^2(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})^2 \middle| \mathcal{F}_{\underline{s}}\right]; \\
&\mathbb{E}\left[c_\Delta^2\left(\widehat{X}_{\underline{s}}\right)\left(Z_s - Z_{\underline{s}}\right)^2 \middle| \mathcal{F}_{\underline{s}}\right]\Big\} \leq C\Delta. \qquad (22)
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\left[-\gamma \widehat{X}_s^2\right] \leq \mathbb{E}\left[-\gamma \widehat{X}_{\underline{s}}^2\right] + C|\gamma|\Delta. \qquad (23)
$$

A similar argument yields to

$$
\mathbb{E}\left[\widehat{X}_s b(\widehat{X}_{\underline{s}})\right] \leq \mathbb{E}\left[\widehat{X}_{\underline{s}} b(\widehat{X}_{\underline{s}})\right] + C\Delta. \qquad (24)
$$

Thanks to Lemma 3.2, the expectation of the stochastic integrals in (21) is equal to zero. It then follows from (19), (21), (23), (24) that

$$
\begin{aligned}
\mathbb{E}\left[e^{-2\gamma t}\widehat{X}_t^2\right] &\leq x_0^2 + 2\int_0^t e^{-2\gamma s}\Big(\mathbb{E}\Big[-\gamma \widehat{X}_{\underline{s}}^2 + \widehat{X}_{\underline{s}} b(\widehat{X}_{\underline{s}}) \\
&\qquad + \frac{1}{2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) + \frac{1}{2}c_\Delta^2(\widehat{X}_{\underline{s}})\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z)\Big] + C\Delta\Big)\mathrm{d}s \\
&\leq x_0^2 + C(\eta+1)\int_0^t e^{-2\gamma s}\mathrm{d}s. \qquad (25)
\end{aligned}
$$

Using the fact that $\widehat{X}_{\underline{t}} = \widehat{X}_t - b\left(\widehat{X}_{\underline{t}}\right)(t-\underline{t}) - \sigma_\Delta\left(\widehat{X}_{\underline{t}}\right)\left(W_t - W_{\underline{t}}\right) - c_\Delta\left(\widehat{X}_{\underline{t}-}\right)\left(Z_t - Z_{\underline{t}}\right)$ together with (22), we get the following estimate for any $p > 1$

$$
\begin{aligned}
\mathbb{E}\left[\left|\widehat{X}_{\underline{t}}\right|^p\right] &\leq 4^{p-1}\Big(\mathbb{E}\left[\left|\widehat{X}_t\right|^p\right] + \mathbb{E}\left[\left|b(\widehat{X}_{\underline{t}})(t-\underline{t})\right|^p\right] + \mathbb{E}\left[\left|\sigma_\Delta(\widehat{X}_{\underline{t}})(W_t - W_{\underline{t}})\right|^p\right] \\
&\qquad + \mathbb{E}\left[\left|c_\Delta(\widehat{X}_{\underline{t}})(Z_t - Z_{\underline{t}})\right|^p\right]\Big) \\
&\leq 4^{p-1}\left(\mathbb{E}\left[\left|\widehat{X}_t\right|^p\right] + C\Delta^p + C\Delta^{p/2} + C\Delta^{1\wedge p/2}\right). \qquad (26)
\end{aligned}
$$

It follows from (25) and (26) that (20) holds for $k=1$.

Second, assume that (20) holds for any $k \leq k_0 \leq [p_0/2]-1$, we will show that (20) still holds for $k = k_0+1$. Here, we use the notation $[p_0/2]$ for the integer part of $p_0/2$.

By applying Itô's formula for $e^{-p\gamma t}\widehat{X}_t^p$ with $p = 2(k_0+1)$ being an even integer, we have

$$
\begin{aligned}
e^{-p\gamma t}\left|\widehat{X}_t\right|^p &= x_0^p + \int_0^t e^{-p\gamma s}\Big[-p\gamma \widehat{X}_s^p + p\widehat{X}_s^{p-1} b(\widehat{X}_{\underline{s}}) + \frac{p(p-1)}{2}\widehat{X}_s^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) \\
&\quad + \int_{\mathbb{R}_0}\Big(\big(\widehat{X}_s + c_\Delta(\widehat{X}_{\underline{s}})z\big)^p - \widehat{X}_s^p - p\widehat{X}_s^{p-1}c_\Delta(\widehat{X}_{\underline{s}})z\Big)\,\nu(\mathrm{d}z)\Big]\,\mathrm{d}s \\
&\quad + p\int_0^t e^{-p\gamma s}\widehat{X}_s^{p-2}\widehat{X}_s\sigma_\Delta(\widehat{X}_{\underline{s}})\mathrm{d}W_s \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-p\gamma s}\Big(\big(\widehat{X}_{s-} + c_\Delta(\widehat{X}_{\underline{s}-})z\big)^p - \widehat{X}_{s-}^p\Big)\widetilde{N}(\mathrm{d}s,\mathrm{d}z). \qquad (27)
\end{aligned}
$$

It follows from (16) and the binomial theorem that for any positive integer q,

$$
\begin{aligned}
\widehat{X}_s^q = &\sum_{0\le i,j,r,v\le q,\, i+j+r+v=q}\frac{q!}{i!j!r!v!}\left(\widehat{X}_{\underline{s}}\right)^i\left(b(\widehat{X}_{\underline{s}})(s-\underline{s})\right)^j \\
&\left(\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})\right)^r\left(c_\Delta(\widehat{X}_{\underline{s}})(Z_s - Z_{\underline{s}})\right)^v. \qquad (28)
\end{aligned}
$$

Using (12), the independence between W and Z, Burkholder–Davis–Gundy's inequality, (12) and **C6**, we have

$$
\begin{aligned}
&\mathbb{E}\left[-\gamma\widehat{X}_s^p\middle|\mathcal{F}_{\underline{s}}\right] \\
&\quad\le -\gamma\widehat{X}_{\underline{s}}^p - p\gamma\widehat{X}_{\underline{s}}b(\widehat{X}_{\underline{s}})(s-\underline{s})\widehat{X}_{\underline{s}}^{p-2} \\
&\qquad + \sum_{0\le i\le p-2,\, i+j+2r=p}\frac{C_p|\gamma|p!}{i!j!r!}\left|\widehat{X}_{\underline{s}}\right|^i\left|b(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^j\left|\sigma_\Delta^2(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^r \\
&\qquad + \sum_{0\le i\le p-2,\, v\ge 2,\, i+j+2r+v=p}\frac{C_p|\gamma|p!}{i!j!r!v!}\left|\widehat{X}_{\underline{s}}\right|^i\left|b(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^j \\
&\qquad \times\left|\sigma_\Delta^2(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^r\left|c_\Delta^v(\widehat{X}_{\underline{s}})(s-\underline{s})\right|.
\end{aligned}
$$

Again, using **T3**, **T4** and (11), we get

$$
\mathbb{E}\left[-\gamma\widehat{X}_s^p\middle|\mathcal{F}_{\underline{s}}\right] \le -\gamma\widehat{X}_{\underline{s}}^p + C_p|\gamma|\sum_{i=0}^{p-2}|\widehat{X}_{\underline{s}}|^i. \qquad (29)
$$

Choosing $q = p-1$ and $q = p-2$ in (28) and using the same argument, we get

$$
\mathbb{E}\left[\widehat{X}_s^{p-1}b(\widehat{X}_{\underline{s}})\middle|\mathcal{F}_{\underline{s}}\right] \le \widehat{X}_{\underline{s}}^{p-1}b(\widehat{X}_{\underline{s}}) + C_p\sum_{i=0}^{p-2}|\widehat{X}_{\underline{s}}|^i \qquad (30)
$$

and

$$
\mathbb{E}\left[\widehat{X}_s^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}})\middle|\mathcal{F}_{\underline{s}}\right] \le \widehat{X}_{\underline{s}}^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) + C_p\sum_{i=0}^{p-2}|\widehat{X}_{\underline{s}}|^i. \qquad (31)
$$

Now, using the binomial theorem, we have

$$
\left(\widehat{X}_s + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right)^p - \widehat{X}_s^p - p\widehat{X}_s^{p-1}c_\Delta\left(\widehat{X}_{\underline{s}}\right)z = \sum_{i=2}^{p}\binom{p}{i}\widehat{X}_s^{p-i}c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)z^i. \qquad (32)
$$

Then, applying (28) to $q = p - j$ with $j \in \{2, \dots, p\}$, we obtain that, for $4 \le i \le p$,

$$\begin{aligned}\mathbb{E}\left[\widehat{X}_s^{p-i} c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)|\mathcal{F}_{\underline{s}}\right] &\le \widehat{X}_{\underline{s}}^{p-i} c_\Delta^i\left(\widehat{X}_{\underline{s}}\right) + (p-i)\widehat{X}_{\underline{s}} b\left(\widehat{X}_{\underline{s}}\right) c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s})\widehat{X}_{\underline{s}}^{p-i-2} \\ &\quad + \mathbb{E}\left[Q_{p-2}\left(\widehat{X}_s\right)|\mathcal{F}_{\underline{s}}\right] c_\Delta^i\left(\widehat{X}_{\underline{s}}\right) \\ &\le \widehat{X}_{\underline{s}}^{p-i} c_\Delta^i\left(\widehat{X}_{\underline{s}}\right) + C_p Q_{p-2}\left(\widehat{X}_{\underline{s}}\right). \end{aligned} \tag{33}$$

Using (32), (33), and **T3**, **C1**, proceeding as in (6) and (7), we obtain that

$$\begin{aligned}&\mathbb{E}\left[\left(\widehat{X}_s + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right)^p - \widehat{X}_s^p - p\widehat{X}_s^{p-1} c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\Big|\mathcal{F}_{\underline{s}}\right] \\ &\le \sum_{i=2}^p \binom{p}{i}\widehat{X}_{\underline{s}}^{p-i} c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)z^i + C Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|, z\right) \\ &\le c_\Delta^2\left(\widehat{X}_{\underline{s}}\right)|\widehat{X}_{\underline{s}}|^{p-2}\frac{p}{2L_0}|z|\left((1+L_0|z|)^{p-1} - 1\right) \\ &\quad + c_\Delta^2\left(\widehat{X}_{\underline{s}}\right)\sum_{i=3}^p \binom{p}{i} L_0^{i-2}\left(\frac{i-2}{2}|\widehat{X}_{\underline{s}}|^{p-4} + \sum_{j=2}^{i-2}\binom{i-2}{j}|\widehat{X}_{\underline{s}}|^{p-2-j}\right)|z|^i \\ &\quad + C Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|, z\right). \end{aligned} \tag{34}$$

Consequently, from (29),(30), (31), (34), **C6**, (19) and Remark 3.4, and $c_\Delta^2(x) \le c^2(x) \le 2L_0^2(1+x^2)$ for any $x \in \mathbb{R}$, we obtain that

$$\begin{aligned}&\mathbb{E}\Bigg[-p\gamma\widehat{X}_s^p + p\widehat{X}_s^{p-1} b(\widehat{X}_{\underline{s}}) + \frac{p(p-1)}{2}\widehat{X}_s^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) \\ &\quad + \int_{\mathbb{R}_0}\left(\left(\widehat{X}_s + c_\Delta(\widehat{X}_{\underline{s}})z\right)^p - \widehat{X}_s^p - p\widehat{X}_s^{p-1} c_\Delta(\widehat{X}_{\underline{s}})z\right)\nu(\mathrm{d}z)\,\Big|\mathcal{F}_{\underline{s}}\Bigg] \\ &\le p|\widehat{X}_{\underline{s}}|^{p-2}\Bigg(-\gamma\widehat{X}_{\underline{s}}^2 + \widehat{X}_{\underline{s}} b\left(\widehat{X}_{\underline{s}}\right) + \frac{p-1}{2}\sigma_\Delta^2\left(\widehat{X}_{\underline{s}}\right) + \frac{c_\Delta^2(\widehat{X}_{\underline{s}})}{2L_0} \\ &\quad \times \int_{\mathbb{R}_0}|z|\left((1+L_0|z|)^{p-1} - 1\right)\nu(\mathrm{d}z)\Bigg) + Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|\right) \\ &\le p\eta|\widehat{X}_{\underline{s}}|^{p-2} + Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|\right). \end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\Bigg[-p\gamma\widehat{X}_s^p + p\widehat{X}_s^{p-1} b(\widehat{X}_{\underline{s}}) + \frac{p(p-1)}{2}\widehat{X}_s^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) \\ &\quad + \int_{\mathbb{R}_0}\left(\left(\widehat{X}_s + c_\Delta(\widehat{X}_{\underline{s}})z\right)^p - \widehat{X}_s^p - p\widehat{X}_s^{p-1} c_\Delta(\widehat{X}_{\underline{s}})z\right)\nu(\mathrm{d}z)\Bigg] \\ &\le C(p, \eta, L_0)\sum_{i=0}^{p-2}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^i\right]. \end{aligned} \tag{35}$$

Thanks to Lemma 3.2, the expectation of the stochastic integrals in (27) is equal to zero. Then, from the estimates (26), (27), (35) and the inductive assumption, we obtain that (20) holds for $k = k_0 + 1$, which implies the desired result. □

Remark 3.5 If $\gamma < 0$, then the approximated solution is stable in the sense that for any $0 \leq p \leq 2[p_0/2]$ there exists a positive constant C, which does not depend on Δ, such that

$$\sup_{t\geq 0} \mathbb{E}\left[|\widehat{X}_t|^p\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}|^p\right] < C.$$

Remark 3.6 Suppose that all conditions of Theorem 3.3 hold, then the bound on the expectation of the number of time steps N_T required by a path approximation on $[0, T]$ for any $T > 0$ is given by

$$\mathbb{E}\left[N_T - 1\right] \leq \frac{C}{\Delta}, \tag{36}$$

where C is a positive constant that does not depend on Δ.

By following the argument used in the proof of Lemma 2 in Fang and Giles (2020), we can obtain the estimate (36) as a consequence of Lemma 3.2 and Theorem 3.3.

The following uniform bound in time for the difference between $\widehat{X}_t$ and $\widehat{X}_{\underline{t}}$ will be required.

Lemma 3.7 *Suppose that coefficients $b, c, \sigma, \sigma_\Delta, c_\Delta$ and the Lévy measure ν satisfy all conditions of Theorem 3.3 and $p \in (0; p_0]$, then there exists a positive constant $C_p = C(p, L)$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left[|\widehat{X}_t - \widehat{X}_{\underline{t}}|^p\right] \leq C_p \Delta^{1\wedge p/2},$$

Proof From (16), for any $p \geq 1$,

$$\begin{aligned}
&|\widehat{X}_t - \widehat{X}_{\underline{t}}|^p \\
&\quad = \left|b(\widehat{X}_{\underline{t}})(t-\underline{t}) + \sigma_\Delta(\widehat{X}_{\underline{t}})(W_t - W_{\underline{t}}) + c_\Delta(\widehat{X}_{\underline{t}})(Z_t - Z_{\underline{t}})\right|^p \\
&\quad \leq 3^{p-1}\left[\left|b(\widehat{X}_{\underline{t}})(t-\underline{t})\right|^p + \left|\sigma_\Delta(\widehat{X}_{\underline{t}})(W_t - W_{\underline{t}})\right|^p + \left|c_\Delta(\widehat{X}_{\underline{t}})(Z_t - Z_{\underline{t}})\right|^p\right] \\
&\quad \leq 3^{p-1}\left[\left|b(\widehat{X}_{\underline{t}})\right|^p \left|h(\widehat{X}_{\underline{t}})\right|^p \Delta^p + \left|\sigma_\Delta(\widehat{X}_{\underline{t}})\right|^p \left|W_t - W_{\underline{t}}\right|^p + \left|c_\Delta(\widehat{X}_{\underline{t}})\right|^p \left|Z_t - Z_{\underline{t}}\right|^p\right].
\end{aligned}$$

By applying **T3** and (11), we have

$$\left|b(\widehat{X}_{\underline{t}})h(\widehat{X}_{\underline{t}})\right| \leq C;\ \sigma_\Delta^2(\widehat{X}_{\underline{t}})h(\widehat{X}_{\underline{t}}) \leq C \text{ and } \left|c_\Delta^p(\widehat{X}_{\underline{t}})h(\widehat{X}_{\underline{t}})^{1\wedge p/2}\right| \leq C,$$

for some positive constant C. Consequently, using Burkholder–Davis–Gundy's inequality, **C6** and (12), we obtain the desired result. For $0 < p < 1$, it suffices to use Hölder's inequality. □

3.3 Convergence of the tamed-adaptive Euler–Maruyama scheme

We consider the following assumptions on the coefficients of Eq. (1).

C7. Coefficient b is one-sided Lipschitz: there exists a constant L_1 such that

$$(x-y)(b(x)-b(y)) \leq L_1|x-y|^2,$$

for any $x, y \in \mathbb{R}$.

C8. Coefficient b is locally Lipschitz continuous: there exist positive constants l and L_2 such that

$$|b(x) - b(y)| \leq L_2 \left(1 + |x|^l + |y|^l\right) |x - y|,$$

for any $x, y \in \mathbb{R}$.

C9. Coefficient σ is $(\alpha + \frac{1}{2})$-locally Hölder continuous: there exist positive constants m, L_3 and $\alpha \in [0, \frac{1}{2}]$ such that

$$|\sigma(x) - \sigma(y)| \leq L_3 \left(1 + |x|^m + |y|^m\right) |x - y|^{1/2+\alpha},$$

for any $x, y \in \mathbb{R}$.

C10. Coefficient c is Lipschitz: there exists a positive constant L_4 such that

$$|c(x) - c(y)| \leq L_4 |x - y|,$$

for any $x, y \in \mathbb{R}$.

Note that if c satisfies Condition **C10** then it also satisfies Condition **C1** with $L_0 = \max\{L_4, |c(0)|\}$. Moreover, conditions **C8, C9, C10** imply conditions **C3, C4, C5**. Therefore, under Conditions **C2**, **C8–C10** and $\int_{\mathbb{R}_0} |z|\nu(\mathrm{d}z) < \infty$, $\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z) < \infty$, Eq. (1) has a unique strong solution.

Remark 3.8 It can be checked that under Conditions **C7–C9** and $\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z) < \infty$, the following functions

$$c_\Delta(x) = \frac{c(x)}{1 + \Delta^{1/2}|c(x)|(1 + |b(x)|)}, \quad \sigma_\Delta(x) = \frac{\sigma(x)}{1 + \Delta^{1/2}|\sigma(x)|} \tag{37}$$

satisfy all conditions of Proposition 3.1.

We are in the position to state the main result of this paper.

Theorem 3.9 *Assume that Conditions **C2, C6–C10** hold and $p_0 \geq \max\{4l; 2 + 4\alpha + 4m\}$. Assume that the functions $c, b, \sigma, c_\Delta, \sigma_\Delta$ and the Lévy measure ν satisfy all conditions of Theorem 3.3, and*

$$|c(x) - c_\Delta(x)| \leq L_5\Delta^{1/2}c^2(x)(1 + |b(x)|), \quad |\sigma(x) - \sigma_\Delta(x)| \leq L_5\Delta^{1/2}\sigma^2(x), \tag{38}$$

for all $x \in \mathbb{R}$ and some constant $L_5 > 0$.

Then, for any $T > 0$, there exists a positive constant $C_T = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta, T)$ such that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|\widehat{X}_t - X_t|\right] \leq \begin{cases} C_T\Delta^\alpha & \text{if } 0 < \alpha \leq \dfrac{1}{2}, \\ \dfrac{C_T}{\log\frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{39}$$

Moreover, let $\mu := \int_{\mathbb{R}_0} |z|\nu(\mathrm{d}z)$ and assume that $L_1 + 2L_4\mu < 0$, $\gamma < 0$, then there exists a positive constant $C = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta)$ which does not depend on T such that

$$\sup_{t\geq 0}\mathbb{E}\left[|\widehat{X}_t - X_t|\right] \leq \begin{cases} C\Delta^\alpha & \text{if } 0 < \alpha \leq \dfrac{1}{2}, \\ \dfrac{C}{\log\frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{40}$$

Remark 3.10 It is straightforward to verify that under Conditions **C2, C6–C10**, the functions c_Δ and σ_Δ defined in (37) satisfy condition (38).

Proof of Theorem 3.9 Put $Y_t = X_t - \widehat{X}_t$. For any $\lambda \in \mathbb{R}$, applying the property **YW3** and Itô's formula for $e^{-\lambda t}\phi_{\delta\varepsilon}(Y_t)$, we get

$$
\begin{aligned}
e^{-\lambda t}|Y_t| &\le e^{-\lambda t}\varepsilon + e^{-\lambda t}\phi_{\delta\varepsilon}(Y_t)\\
&= e^{-\lambda t}\varepsilon + \int_0^t e^{-\lambda s}\Big[-\lambda\phi_{\delta\varepsilon}(Y_s) + \phi'_{\delta\varepsilon}(Y_s)\left(b(X_s) - b(\widehat{X}_{\underline{s}})\right)\\
&\quad + \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma(X_s) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2\Big]\mathrm{d}s\\
&\quad + \int_0^t e^{-\lambda s}\phi'_{\delta\varepsilon}(Y_s)\left(\sigma(X_s) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right)\mathrm{d}W_s\\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-\lambda s}\left[\phi_{\delta\varepsilon}\left(Y_s + \left(c(X_s) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(Y_s\right)\right.\\
&\quad \left. - \phi'_{\delta\varepsilon}\left(Y_s\right)\left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right]\nu(\mathrm{d}z)\mathrm{d}s\\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-\lambda s}\left[\phi_{\delta\varepsilon}\left(Y_{s-} + \left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(Y_{s-}\right)\right]\widetilde{N}\left(\mathrm{d}s, \mathrm{d}z\right).
\end{aligned}
\tag{41}
$$

Set

$$
\begin{aligned}
J_1(s) &= \phi'_{\delta\varepsilon}(Y_s)\left(b(X_s) - b(\widehat{X}_{\underline{s}})\right),\\
J_2(s) &= \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma(X_s) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2,\\
J_3(s) &= \phi_{\delta\varepsilon}\left(Y_s + \left(c(X_s) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(Y_s\right) - \phi'_{\delta\varepsilon}\left(Y_s\right)\left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z.
\end{aligned}
$$

First, using properties **YW1, YW2**, Conditions **C7, C8** and Cauchy's inequality, we have

$$
\begin{aligned}
J_1(s) &\le \frac{\phi'_{\delta\varepsilon}(|Y_s|)}{|Y_s|}Y_s\left(b(X_s) - b(\widehat{X}_s)\right) + \left|\phi'_{\delta\varepsilon}(Y_s)\left(b(\widehat{X}_s) - b(\widehat{X}_{\underline{s}})\right)\right|\\
&\le L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + L_2\left(1 + |\widehat{X}_s|^l + |\widehat{X}_{\underline{s}}|^l\right)|\widehat{X}_s - \widehat{X}_{\underline{s}}|\\
&\le L_1|Y_s| + \frac{3}{2}L_2\Delta^{1/2}\left(1 + |\widehat{X}_s|^{2l} + |\widehat{X}_{\underline{s}}|^{2l}\right) + \frac{1}{2}L_2\Delta^{-1/2}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^2.
\end{aligned}
\tag{42}
$$

Second, using the property **YW5**, the Condition **C9** and (38), we have

$$
\begin{aligned}
J_2(s) &= \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma(X_s) - \sigma(\widehat{X}_s) + \sigma(\widehat{X}_s) - \sigma(\widehat{X}_{\underline{s}}) + \sigma(\widehat{X}_{\underline{s}}) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2\\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\left(\left|\sigma(X_s) - \sigma(\widehat{X}_s)\right|^2 + \left|\sigma(\widehat{X}_s) - \sigma(\widehat{X}_{\underline{s}})\right|^2 + \left|\sigma(\widehat{X}_{\underline{s}}) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2\right)\\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\Big[L_3^2\left(1 + |X_s|^m + |\widehat{X}_s|^m\right)^2|X_s - \widehat{X}_s|^{1+2\alpha}\\
&\quad + L_3^2\left(1 + |\widehat{X}_s|^m + |\widehat{X}_{\underline{s}}|^m\right)^2|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha} + L_5^2\Delta\left|\sigma(\widehat{X}_{\underline{s}})\right|^4\Big]\\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\Big[3L_3^2\left(1 + |X_s|^{2m} + |\widehat{X}_s|^{2m}\right)|Y_s|^{1+2\alpha} +\\
&\quad + 3L_3^2\left(1 + |\widehat{X}_s|^{2m} + |\widehat{X}_{\underline{s}}|^{2m}\right)|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha} + L_5^2\Delta\left|\sigma(\widehat{X}_{\underline{s}})\right|^4\Big]
\end{aligned}
$$

$$\leq \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+|X_s|^{2m}+|\widehat{X}_s|^{2m}\right)+\frac{9L_3^2\delta}{\varepsilon\log\delta}\left(1+C|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{2m}+C|\widehat{X}_{\underline{s}}|^{2m}\right)|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}$$
$$+\frac{3L_5^2\delta\Delta|\sigma(\widehat{X}_{\underline{s}})|^4}{\varepsilon\log\delta}$$
$$\leq \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+|X_s|^{2m}+|\widehat{X}_s|^{2m}\right)+\frac{9L_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}+\frac{9CL_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha+2m}$$
$$+\frac{9CL_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}+\frac{C\delta\Delta\left(|\widehat{X}_{\underline{s}}|^{2+4\alpha+4m}+1\right)}{\varepsilon\log\delta}. \tag{43}$$

Third, using the mean value theorem, the property **YW2**, the Conditions **C1, C10** and (38), there exists an $(\mathcal{F}_s)$-adapted process $\xi=(\xi_s)$ such that

$$\begin{aligned}
J_3(s)&=\phi_{\delta\varepsilon}\left(Y_s+\left(c\left(X_s\right)-c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right)-\phi_{\delta\varepsilon}\left(Y_s\right)-\phi_{\delta\varepsilon}'\left(Y_s\right)\left(c\left(X_s\right)-c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\\
&=\phi_{\delta\varepsilon}'\left(\xi_s\right)\left(c\left(X_s\right)-c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z+\phi_{\delta\varepsilon}'\left(Y_s\right)\left(c\left(X_s\right)-c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\\
&\leq 2\left|c\left(X_s\right)-c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right||z|\\
&\leq 2\left[\left|c\left(X_s\right)-c\left(\widehat{X}_s\right)\right|+\left|c\left(\widehat{X}_s\right)-c\left(\widehat{X}_{\underline{s}}\right)\right|+\left|c\left(\widehat{X}_{\underline{s}}\right)-c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|\right]|z|\\
&\leq 2\left[L_4\left|Y_s\right|+L_4\left|\widehat{X}_s-\widehat{X}_{\underline{s}}\right|+L_5\Delta^{1/2}c^2(\widehat{X}_{\underline{s}})(1+|b(\widehat{X}_{\underline{s}})|)\right]|z|\\
&\leq 2\left[L_4\left|Y_s\right|+L_4\left|\widehat{X}_s-\widehat{X}_{\underline{s}}\right|+C\Delta^{1/2}(1+|\widehat{X}_{\underline{s}}|^{l+3})\right]|z|.
\end{aligned} \tag{44}$$

By choosing $\lambda=L_1+2L_4\mu$ where recall that $\mu=\int_{\mathbb{R}_0}|z|\nu(\mathrm{d}z)$ and using **YW3**, we get

$$(L_1+2L_4\mu)\left[|Y_s|-\phi_{\delta\varepsilon}\left(Y_s\right)\right]\leq\varepsilon\left(|L_1|+2L_4\mu\right). \tag{45}$$

A combination of (41), (42), (43), (44) and (45) implies

$$\begin{aligned}
&\mathbb{E}\left[e^{-(L_1+2L_4\mu)t}|Y_t|\right]\\
&\leq e^{-(L_1+2L_4\mu)t}\varepsilon+\int_0^t e^{-(L_1+2L_4\mu)s}\\
&\left[\varepsilon\left(|L_1|+2L_4\right)+\frac{3}{2}L_2\Delta^{1/2}\left(1+\mathbb{E}\left[|\widehat{X}_s|^{2l}\right]+\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2l}\right]\right)\right.\\
&+\frac{1}{2}L_2\Delta^{-1/2}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^2\right]+\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+\mathbb{E}\left[|X_s|^{2m}\right]+\mathbb{E}\left[|\widehat{X}_s|^{2m}\right]\right)\\
&+\frac{9L_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}\right]+\frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha+2m}\right]\\
&\left.+\frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}\right]+\frac{C\delta\Delta\left(\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2+4\alpha+4m}\right]+1\right)}{\varepsilon\log\delta}\right]\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{R}_0}2e^{-(L_1+2L_4\mu)s}\left(L_4\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|\right]+C\Delta^{1/2}\left(1+\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{l+3}\right]\right)\right)|z|\nu(\mathrm{d}z)\mathrm{d}s\\
&\leq e^{-(L_1+2L_4\mu)t}\varepsilon+\int_0^t e^{-(L_1+2L_4\mu)s}\left[\varepsilon\left(|L_1|+2L_4\right)+\frac{3}{2}L_2\Delta^{1/2}\left(1+\mathbb{E}\left[|\widehat{X}_s|^{2l}\right]+\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2l}\right]\right)\right.\\
&+\frac{1}{2}L_2\Delta^{-1/2}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^2\right]+\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+\mathbb{E}\left[|X_s|^{2m}\right]+\mathbb{E}\left[|\widehat{X}_s|^{2m}\right]\right)\\
&+\frac{9L_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}\right]+\frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha+2m}\right]\\
&+\frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}\right]+\frac{C\delta\Delta\left(\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2+4\alpha+4m}\right]+1\right)}{\varepsilon\log\delta}
\end{aligned}$$

$$+2\mu\left(L_4\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|\right]+C\Delta^{1/2}\left(1+\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{l+3}\right]\right)\right)\Big]\mathrm{d}s. \tag{46}$$

Now, using Eq. (16), Burkholder–Davis–Gundy's inequality, **T3**, **C6**, (12), and (11), we obtain

$$\begin{aligned}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]&\leq 3^{2\alpha}\Big(\mathbb{E}\left[|b(\widehat{X}_{\underline{s}})(s-\underline{s})|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]+\mathbb{E}\left[|\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s-W_{\underline{s}})|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]\\&\quad+\mathbb{E}\left[|c_\Delta(\widehat{X}_{\underline{s}})(Z_s-Z_{\underline{s}})|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]\Big)\leq C\Big(|b(\widehat{X}_{\underline{s}})(s-\underline{s})|^{1+2\alpha}+|\sigma_\Delta(\widehat{X}_{\underline{s}})|^{1+2\alpha}(s-\underline{s})^{1/2+\alpha}\\&\quad+|c_\Delta(\widehat{X}_{\underline{s}})|^{1+2\alpha}(s-\underline{s})^{1/2+\alpha}\Big)\\&\leq C\Big(\left|b(\widehat{X}_{\underline{s}})\right|^{1+2\alpha}\left|h(\widehat{X}_{\underline{s}})\right|^{1+2\alpha}\Delta^{1+2\alpha}+|\sigma_\Delta(\widehat{X}_{\underline{s}})|^{1+2\alpha}\left|h(\widehat{X}_{\underline{s}})\right|^{1/2+\alpha}\Delta^{1/2+\alpha}\\&\quad+\left|c_\Delta(\widehat{X}_{\underline{s}})\right|^{1+2\alpha}\left|h(\widehat{X}_{\underline{s}})\right|^{1/2+\alpha}\Delta^{1/2+\alpha}\Big)\leq C\Delta^{1/2+\alpha}.\end{aligned}$$

This, together with Theorem 3.3 and $m\leq p_0/2$, yields that

$$\begin{aligned}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}\right]&=\mathbb{E}\left[\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]\right]\\&=\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]\right]\\&\leq C\Delta^{1/2+\alpha}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}\right]\leq C\Delta^{1/2+\alpha}.\end{aligned} \tag{47}$$

Consequently, plugging (47) into (46), and using condition $p_0\geq\max\{4l;2+4\alpha+4m\}$, Theorem 3.3, Proposition 2.3, and Lemma 3.7, for any $T>0$, there exists a positive constant C_T such that for any $t\in[0,T]$,

$$\begin{aligned}\mathbb{E}\left[e^{-(L_1+2L_4\mu)t}|Y_t|\right]&\leq e^{-(L_1+2L_4\mu)t}\varepsilon+C_T\\&\left[\varepsilon+\Delta^{1/2}+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta\Delta^{1/2+\alpha}}{\varepsilon\log\delta}+\frac{\delta\Delta}{\varepsilon\log\delta}\right]\int_0^t e^{-(L_1+2L_4\mu)s}\mathrm{d}s.\end{aligned} \tag{48}$$

If $\alpha\in\left(0;\frac{1}{2}\right]$, choosing $\varepsilon=\Delta^{1/2}$, $\delta=2$, we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\left[|Y_t|\right]\leq C_T\Delta^{\alpha}.$$

If $\alpha=0$, choosing $\varepsilon=\Delta^{1/4}$, $\delta=\Delta^{-1/4}$, we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\left[|Y_t|\right]\leq\frac{C_T}{\log\frac{1}{\Delta}}.$$

Therefore, we have shown (39). Note that if $L_1+2L_4\mu<0$ and $\gamma<0$, we can choose the constant C_T in (48) such that it does not depend on T. Therefore, we also obtain (40). This finishes the proof. □

4 Numerical experiments

We consider numerical experiments for four different SDEs with coefficients given in Table 1. For all equations, $X_0=0$, $(Z_t)_{t\geq0}$ is a compound Poisson process of the form $Z_t=\sum_{i=1}^{N_t}\xi_i$, where $(N_t)_{t\geq0}$ is a Poisson process with intensity $\lambda=5$, and $(\xi_i)_{i\geq1}$ is a sequence of independent and identically distributed random variables. We suppose that each ξ_i has a normal distribution with mean zero and standard deviation 0.2. A simple computation shows that these equations satisfy Conditions **C2**, **C6–C10** with constants $p_0, L_1, \gamma, l, m, \alpha$ given in Table 1. In all these cases, $p_0\geq\max\{4l;2+4\alpha+4m\}$, hence it follows from Theorem

Table 1 Four jump SDEs with their parameters

Case	b	σ	c	p_0	L_1	γ	η	l	m	α
1	$-1+x-x^3$	$1+(1+x)x^{2/3}$	$x+\sin(x)$	10	1	-1	31,873	2	$\frac{4}{3}$	$\frac{1}{6}$
2	$-1+x-x^3$	$1+\sqrt{\frac{x^4+x^{4/3}}{14}}$	$x+\sin(x)$	10	1	1	957	2	2	$\frac{1}{6}$
3	$-1-x-x^{7/3}$	$1+\sqrt{\frac{2x^2+x^{10/3}+x^{4/3}}{14}}$	$x+\sin(x)$	10	-1	1	1868	$\frac{4}{3}$	1	$\frac{1}{6}$
4	$-1-x-x^{7/3}$	$1+\sqrt{\frac{x^{10/3}+x^{4/3}}{14}}$	$x+\sin(x)$	10	-1	-1	1583	$\frac{4}{3}$	1	$\frac{1}{6}$

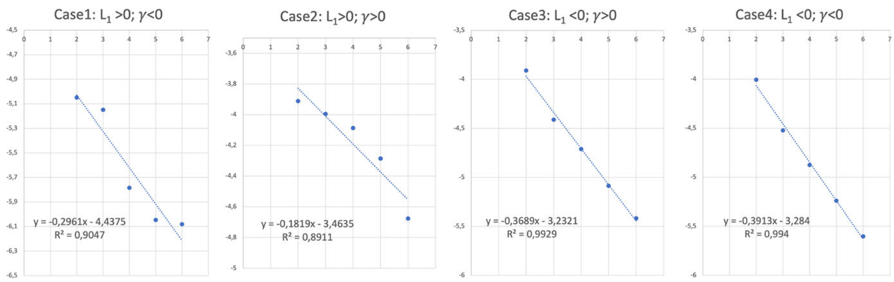

Fig. 1 Values of $\log_2(me(l))$ for $l = 2, 3, 4, 5, 6$

3.9 that the tamed-adaptive Euler–Maruyama approximation scheme defined by (10), (11) and (37) converges in L^1-norm at the rate of order α in any finite time interval. Moreover, in Case 4, since $L_1 < 0$ and $\gamma < 0$, the tamed-adaptive Euler–Maruyama approximation scheme (10) converges in L^1-norm at the rate of order α in infinite time intervals.

To study the empirical rates of convergence of the tamed-adaptive Euler–Maruyama scheme, we consider

$$me(l) = \frac{1}{M}\sum_{k=1}^{M} |\widehat{X}_5^{(l,k)} - \widehat{X}_5^{(l+1,k)}|,$$

where for each $l \geq 2$, $(\widehat{X}^{(l,k)})_{1\leq k\leq M}$ is a sequence of independent copies of $\widehat{X}^{(l)}$ defined by Eqs. (10), (11), and (37) with $\Delta = 2^{-l}$. We adapt the Algorithm 1 in Fang and Giles (2020) to generate $\widehat{X}_5^{(l,k)}$ and $\widehat{X}_5^{(l+1,k)}$ on the same Brownian motion W and Lévy process Z for each k and l.

If $\widehat{X}^{(l)}$ converges at the rate of order $\beta \in (0, +\infty)$ in L^1-norm, then there exists a constant $\beta > 0$ such that $2^{\beta l}\mathbb{E}[|X_5 - X_5^{(l)}|] = O(1)$, implying that $2^{\beta l}\mathbb{E}[|\widehat{X}_5^{(l+1)} - \widehat{X}_5^{(l)}|] = O(1)$ and vice-versa. In this case, we can write $\log_2 me(l) = -\beta l + C + o(1)$, for some constant $C \in \mathbb{R}$. Thus, β can be estimated by the regression method.

Figure 1 shows the simulation result of $\log_2 me(l)$ for $l = 2, \ldots, 6$. We draw the regression lines to estimate the empirical rates of convergence β in each case. In Case 2, the empirical rate of convergence, which is 0.1819, is almost the same as the theoretical rate, which is 1/6. In the other cases, the empirical rates are slightly better than the theoretical rate.

Note that in Case 4, the tamed-adaptive Euler–Maruyama approximation converges in infinite time intervals while in other cases, it converges in any finite time intervals.

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5 Appendix

In this appendix, we will prove the theorem Theorem 2.1. Without loss of generality, we assume that γ is a positive constant.

5.1 Existence of solution

For each $N > 0$, set

$$b_N(x) = \begin{cases} b(x), & \text{if } |x| \le N, \\ b\left(\frac{Nx}{|x|}\right)(N+1-|x|), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \ge N+1, \end{cases}$$

and

$$\sigma_N(x) = \begin{cases} \sigma(x), & \text{if } |x| \le N, \\ \sigma\left(\frac{Nx}{|x|}\right)(N+1-|x|), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \ge N+1, \end{cases}$$

and

$$c_N(x) = \begin{cases} c(x), & \text{if } |x| \le N, \\ c\left(\frac{Nx}{|x|}\right)(N+1-|x|), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \ge N+1. \end{cases}$$

It is clearly that b_N, c_N and σ_N satisfy Assumptions of Theorem 2.2 in Li and Mytnik (2011). Thus, the equation

$$X_t^N = x_0 + \int_0^t b_N\left(X_s^N\right) \mathrm{d}s + \int_0^t \sigma_N\left(X_s^N\right) \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}_0} c_N\left(X_{s-}^N\right) z\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \quad (49)$$

has a unique strong solution X_t^N. We will show that when $N \to \infty$, X_t^N converges in probability to a process X_t which satisfies Eq. (1).

For each $N > 0$, set

$$\tau_N = T \wedge \inf\left\{t \in [0; T] : \left|X_t^N\right| \ge N\right\}.$$

Due to the pathwise uniqueness of solution to Eq. (49), $X_t^N = X_t^M$ almost surely for any $t < \tau_N$ and $N < M$. Then, we will show that $\tau_N = T$ almost surely for all N large enough.

It is straightforward to show that the coefficients $b_N(x)$, $c_N(x)$ and $\sigma_N(x)$ satisfy Condition **C2'**:

$$p_0 x b(x) + \frac{p_0(p_0-1)}{2}\sigma^2(x) + \frac{c^2(x)}{4L_0^2}\int_{\mathbb{R}_0}\left((1+2L_0|z|)^{p_0} - 1 - 2p_0L_0|z|\right)\nu(\mathrm{d}z)$$

$$\leq 2\gamma |x|^2 + 2\eta,$$

for any $x \in \mathbb{R}$.

Secondly, for any $p \in (1; 2)$, using (55) to get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^N\right|^p\right] \leq C \quad \text{for any } N > 0.$$

Thus,

$$C \geq \mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^N\right|^p\right] \geq \mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^N\right|^p \mathbb{I}_{[\tau_N < T]}\right] \geq N^p \mathbb{P}\left[\tau_N < T\right].$$

It leads to $\sum_{N=1}^{\infty} \mathbb{P}\left(\tau_N < T\right) < \infty$. Thanks to Borel–Cantelli's lemma, we obtain

$$\mathbb{P}\left[\limsup_N \{\tau_N < T\}\right] = 0.$$

Since $(\tau_N)_N$ is increasing, $\tau_N = T$ for all N large enough. It means $\lim_{N\to\infty} X_t^N = X_t$ exists almost surely and $X_t = X_t^M$ almost surely for any $t < \tau_N$ and $M \geq N$. On the other hand, for any $\kappa > 0$, $k > 1$, $2 < q \leq p_0$,

$$\begin{aligned}\mathbb{E}\left[\left|X_{t\wedge\tau_{N+k}}^{N+k} - X_{t\wedge\tau_N}^N\right|^2\right] &\leq 2\mathbb{E}\left[\left(\left|X_t^{N+k}\right|^2 + \left|X_t^N\right|^2\right)\mathbb{I}_{\{\tau_N < T\}}\right]\\ &\leq C_q\left(\kappa\mathbb{E}\left[\left|X_t^{N+k}\right|^q\right] + \kappa\mathbb{E}\left[\left|X_t^N\right|^q\right] + \frac{\mathbb{P}\left[\tau_N < T\right]}{\kappa^{2/(q-2)}}\right).\end{aligned}$$

First, let $N \to \infty$ and then let $\kappa \to 0$, we get

$$\mathbb{E}\left[\left|X_{t\wedge\tau_{N+M}}^{N+M} - X_{t\wedge\tau_N}^N\right|^2\right] \to 0 \quad \text{as } N \to \infty. \tag{50}$$

It means that $(X_{t\wedge\tau_N}^N)_{N\geq 1}$ is a Cauchy sequence in L^2 space. This implies

$$X_{t\wedge\tau_N}^N \xrightarrow{L^2} X_t \quad \text{as } N \to \infty.$$

Furthermore, for any $p \in (0; p_0]$, since coefficients b_N, c_N and σ_N satisfy Condition **C2'**, there exists a constant $C_p > 0$ such that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|X_t^N|^p\right] \leq C_p.$$

Thanks to Fatou's lemma, there exists a constant $C_p > 0$ such that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|X_t|^p\right] \leq C_p.$$

From the definition of $b_N(x)$, we have

$$\mathbb{E}\left[\left|\int_0^{t\wedge\tau_N}\left[b_N\left(X_s^N\right) - b(X_s)\right]\mathrm{d}s\right|^2\right] = 0.$$

Moreover, using the fact that $|b(x)| \leq C\left(1 + |x|^{\ell}\right)$ for all $x \in \mathbb{R}$ and $p_0 \geq 4l$,

$$\mathbb{E}\left[\left|\int_{t\wedge\tau_N}^t b(X_s)ds\right|^2\right] \le C\int_0^t \mathbb{E}\left[\left(1+|X_s|^{2\ell}\right)\mathbb{I}_{\{\tau_N\le s\}}\right]\mathrm{d}s$$
$$\le \kappa C\int_0^T \mathbb{E}\left[1+|X_s|^{2\ell}\right]^2\mathrm{d}s + \frac{C\mathbb{P}[\tau_N<T]}{\kappa}$$
$$\le \kappa C + \frac{C\mathbb{P}[\tau_N<T]}{\kappa}.$$

Let $N\to\infty$ and $\kappa\to 0$, we have

$$\int_0^{t\wedge\tau_N} b_N\left(X_s^N\right)\mathrm{d}s \xrightarrow{L^2} \int_0^t b(X_s)\mathrm{d}s \quad \text{as } N\to\infty. \tag{51}$$

In the same manner, we can see that

$$\int_0^{t\wedge\tau_N} \sigma_N\left(X_s^N\right)\mathrm{d}W_s \xrightarrow{L^2} \int_0^t \sigma(X_s)\mathrm{d}W_s \quad \text{as } N\to\infty, \tag{52}$$

and, by $\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z)<\infty$,

$$\int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0} c_N\left(X_{s-}^N\right)z\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \xrightarrow{L^2} \int_0^t\int_{\mathbb{R}_0} c\left(X_{s-}\right)z\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \quad \text{as } N\to\infty. \tag{53}$$

By combining (49), (50), (51), (52) and (53), we get

$$X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s + \int_0^t\int_{\mathbb{R}_0} c\left(X_{s-}\right)z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$$

almost surely for all $t\in[0,T]$. This shows that $(X_t)_{t\in[0,T]}$ is a solution of Eq. (1). The proof is complete.

5.2 Pathwise uniqueness

Suppose that Eq. (1) has a solution $(X_t)_{0\le t\le T}$, and $(X'_t)_{0\le t\le T}$ is another solution of Eq. (1). It follows from the proof of Proposition 2.3 that the sample paths of $(X_t)_{0\le t\le T}$ and $(X'_t)_{0\le t\le T}$ do not explode. We are going to show that $\mathbb{E}\left[|X_t-X'_t|\right]=0$ for all $t\in[0,T]$, which implies the uniqueness of solution. For each $N>0$, let $\tau_N = T\wedge\inf\left\{t\ge 0: |X_t|\vee\left|X'_t\right|\ge N\right\}$.

Firstly, applying Itô's formula for X_t^2 and Condition **C2** with $p_0=2$, we have

$$\begin{aligned}
X_t^2 &= x_0^2 + \int_0^t \left(2X_sb\left(X_s\right)+\sigma^2\left(X_s\right)\right)\mathrm{d}s + \int_0^t 2X_s\sigma\left(X_s\right)\mathrm{d}W_s\\
&\quad + \int_0^t\int_{\mathbb{R}_0}\left(\left(X_s+c\left(X_s\right)z\right)^2 - X_s^2 - 2X_sc\left(X_s\right)z\right)\nu(\mathrm{d}z)\mathrm{d}s\\
&\quad + \int_0^t\int_{\mathbb{R}_0}\left(\left(X_{s-}+c\left(X_{s-}\right)z\right)^2 - X_{s-}^2\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&= x_0^2 + \int_0^t \left(2X_sb\left(X_s\right)+\sigma^2\left(X_s\right)\right)\mathrm{d}s + \int_0^t 2X_s\sigma\left(X_s\right)\mathrm{d}W_s\\
&\quad + \int_0^t\int_{\mathbb{R}_0} c^2\left(X_s\right)z^2\nu\left(\mathrm{d}z\right)\mathrm{d}s + \int_0^t\int_{\mathbb{R}_0}\left(2X_{s-}c\left(X_{s-}\right)z + c^2\left(X_{s-}\right)z^2\right)\widetilde{N}\left(\mathrm{d}s,\mathrm{d}z\right)\\
&= x_0^2 + \int_0^t\left(2X_sb\left(X_s\right)+\sigma^2\left(X_s\right)+c^2\left(X_s\right)\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z)\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t 2X_s\sigma\left(X_s\right)\mathrm{d}W_s+\int_0^t\int_{\mathbb{R}_0}\left(2X_{s-}c\left(X_{s-}\right)z+c^2\left(X_{s-}\right)z^2\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&\leq x_0^2+2\int_0^t\left(\gamma\left|X_s\right|^2+\eta\right)\mathrm{d}s+\int_0^t 2X_s\sigma\left(X_s\right)\mathrm{d}W_s\\
&+\int_0^t\int_{\mathbb{R}_0}\left(2X_{s-}c\left(X_{s-}\right)z+c^2\left(X_{s-}\right)z^2\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z),
\end{aligned}
$$

which implies

$$
\begin{aligned}
X_{t\wedge\tau_N}^2\leq x_0^2&+2\int_0^{t\wedge\tau_N}\left(\gamma\left|X_s\right|^2+\eta\right)\mathrm{d}s+\int_0^{t\wedge\tau_N}2X_s\sigma\left(X_s\right)\mathrm{d}W_s\\
&+\int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\left(2X_{s-}c\left(X_{s-}\right)z+c^2\left(X_{s-}\right)z^2\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
$$

For any stopping time $\tau\leq T$, we get

$$
\begin{aligned}
X_{\tau\wedge\tau_N}^2\leq x_0^2&+2\int_0^{\tau\wedge\tau_N}\left(\gamma\left|X_s\right|^2+\eta\right)\mathrm{d}s+\int_0^{\tau\wedge\tau_N}2X_s\sigma\left(X_s\right)\mathrm{d}W_s\\
&+\int_0^{\tau\wedge\tau_N}\int_{\mathbb{R}_0}\left(2X_{s-}c\left(X_{s-}\right)z+c^2\left(X_{s-}\right)z^2\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
\tag{54}
$$

Taking expectation on both sides (54) and using Proposition 2.3 to obtain

$$
\begin{aligned}
\mathbb{E}\left[X_{\tau\wedge\tau_N}^2\right]&\leq x_0^2+2\mathbb{E}\left[\int_0^{\tau\wedge\tau_N}\left(\gamma\left|X_s\right|^2+\eta\right)\mathrm{d}s\right]\\
&\leq x_0^2+2\int_0^T\left(\gamma\mathbb{E}\left[\left|X_s\right|^2\right]+\eta\right)\mathrm{d}s\\
&\leq C(x_0,\gamma,\eta,T).
\end{aligned}
$$

For any $p\in(0;2)$, thanks to Proposition IV.4.7 in Revuz and Yor (1999), we have

$$
\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_{t\wedge\tau_N}\right|^p\right]\leq C(x_0,\gamma,\eta,p,T).
$$

Let $N\to\infty$ and apply Fatou's lemma, we get

$$
\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t\right|^p\right]\leq C(x_0,\gamma,\eta,p,T).
\tag{55}
$$

Similarly, since $(X_t')_{0\leq t\leq T}$ is another solution of Eq. (1), as in (55), for any $p\in(0;2)$, we have

$$
\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t'\right|^p\right]\leq C\left(x_0,\gamma,\eta,p,T\right).
\tag{56}
$$

Since X_t and X_t' are solutions of (1), we write

$$
\begin{aligned}
X_t-X_t'=&\int_0^t\left[b\left(X_s\right)-b\left(X_s'\right)\right]\mathrm{d}s+\int_0^t\left[\sigma\left(X_s\right)-\sigma\left(X_s'\right)\right]\mathrm{d}W_s\\
&+\int_0^t\int_{\mathbb{R}_0}\left(c\left(X_{s-}\right)-c\left(X_{s-}'\right)\right)z\widetilde{N}\left(\mathrm{d}s,\mathrm{d}z\right).
\end{aligned}
$$

Applying Itô's formula for $\phi_{\delta\varepsilon}\left(X_t - X'_t\right)$ and using the mean value theorem, **YW2** and **YW5**, we get

$$
\begin{aligned}
&\left|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right| \\
&\quad \leq \varepsilon + \phi_{\delta\varepsilon}\left(X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right) \\
&\quad = \varepsilon + \int_0^{t\wedge\tau_N} \phi'_{\delta\varepsilon}\left(X_s - X'_s\right)\left(b\left(X_s\right) - b\left(X'_s\right)\right)\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N} \frac{1}{2}\phi''_{\delta\varepsilon}\left(X_s - X'_s\right)\left(\sigma\left(X_s\right) - \sigma\left(X'_s\right)\right)^2\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N} \phi'_{\delta\varepsilon}\left(X_s - X'_s\right)\left(\sigma\left(X_s\right) - \sigma\left(X'_s\right)\right)\mathrm{d}W_s \\
&\qquad + \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}\left(X_s - X'_s + \left(c\left(X_s\right) - c\left(X'_s\right)\right)z\right)\right. \\
&\qquad \left. -\phi_{\delta\varepsilon}\left(X_s - X'_s\right) - \phi'_{\delta\varepsilon}\left(X_s - X'_s\right)\left(c\left(X_s\right) - c\left(X'_s\right)z\right)\right]\nu(\mathrm{d}z)\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-} + \left(c\left(X_{s-}\right) - c\left(X'_{s-}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-}\right)\right]\widetilde{N}\left(\mathrm{d}s, \mathrm{d}z\right) \\
&\quad \leq \varepsilon + \int_0^{t\wedge\tau_N}\left|b\left(X_s\right) - b\left(X'_s\right)\right|\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N}\frac{1}{\left|X_s - X'_s\right|\log\delta}\mathbb{I}_{\left[\frac{\varepsilon}{\delta},\varepsilon\right]}\left(\left|X_s - X'_s\right|\right)\left(\sigma\left(X_s\right) - \sigma\left(X'_s\right)\right)^2\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N} \phi'_{\delta\varepsilon}\left(X_s - X'_s\right)\left(\sigma\left(X_s\right) - \sigma\left(X'_s\right)\right)\mathrm{d}W_s + \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0} 2\left|c\left(X_s\right) - c\left(X'_s\right)\right||z|\nu\left(\mathrm{d}z\right)\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-} + \left(c\left(X_{s-}\right) - c\left(X'_{s-}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-}\right)\right]\widetilde{N}\left(\mathrm{d}s, \mathrm{d}z\right).
\end{aligned}
$$

Then, using Conditions **C3, C4, C5**, we get

$$
\begin{aligned}
&\left|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right| \\
&\quad \leq \varepsilon + \int_0^{t\wedge\tau_N} L_N\left|X_s - X'_s\right|\mathrm{d}s + \int_0^{t\wedge\tau_N}\frac{\varepsilon^{2\alpha}L_N^2}{\log\delta}\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N} \phi'_{\delta\varepsilon}\left(X_s - X'_s\right)\left(\sigma\left(X_s\right) - \sigma\left(X'_s\right)\right)\mathrm{d}W_s + \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0} 2L_N\left|X_s - X'_s\right||z|\nu(\mathrm{d}z)\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-} + \left(c\left(X_{s-}\right) - c\left(X'_{s-}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-}\right)\right]\widetilde{N}\left(\mathrm{d}s, \mathrm{d}z\right) \\
&\quad \leq \varepsilon + \int_0^{t\wedge\tau_N} L_N\left(1 + 2\mu\right)\left|X_s - X'_s\right|\mathrm{d}s + \int_0^{t\wedge\tau_N}\frac{\varepsilon^{2\alpha}L_N^2}{\log\delta}\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N} \phi'_{\delta\varepsilon}\left(X_s - X'_s\right)\left(\sigma\left(X_s\right) - \sigma\left(X'_s\right)\right)\mathrm{d}W_s \\
&\qquad + \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-} + \left(c\left(X_{s-}\right) - c\left(X'_{s-}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(X_{s-} - X'_{s-}\right)\right]\widetilde{N}\left(\mathrm{d}s, \mathrm{d}z\right).
\end{aligned}
\tag{57}
$$

By taking expectation on both sides (57) and using Proposition 2.3, we obtain

$$
\mathbb{E}\left[\left|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right]\right. \leq \varepsilon + L_N\left(1 + 2\mu\right)\int_0^t \mathbb{E}\left[\left|X_{s\wedge\tau_N} - X'_{s\wedge\tau_N}\right|\right]\mathrm{d}s + \frac{\varepsilon^{2\alpha}L_N^2 T}{\log\delta}.
$$

By choosing $\delta = 2$ and letting $\varepsilon \to 0$, we get

$$\mathbb{E}\left[\left|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right|\right] \leq L_N(1+2\mu)\int_0^t \mathbb{E}\left[\left|X_{s\wedge\tau_N} - X'_{s\wedge\tau_N}\right|\right]\mathrm{d}s.$$

Thanks to Gronwall's inequality, $\mathbb{E}\left[\left|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right|\right] = 0$. It means $X_{t\wedge\tau_N} = X'_{t\wedge\tau_N}$ almost surely. This leads to $\mathbb{E}\left[\left|X_t - X'_t\right|\right] = \mathbb{E}\left[\left|X_t - X'_t\right|\mathbb{I}_{[\tau_N \leq t]}\right]$. By applying Cauchy's inequality, (55) and (56) for any $p \in (1; 2)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left|X_t - X'_t\right|\right] &\leq \frac{1}{2N}\mathbb{E}\left[\left|X_t - X'_t\right|^2\right] + \frac{N}{2}\mathbb{P}\left[\tau_N \leq T\right] \\
&\leq \frac{1}{2N}\mathbb{E}\left[\left|X_t - X'_t\right|^2\right] + \frac{1}{2N^{p-1}}\left(\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|^p\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\left|X'_t\right|^p\right]\right) \\
&\leq \frac{1}{2N}\mathbb{E}\left[\left|X_t - X'_t\right|^2\right] + \frac{C}{2N^{p-1}}.
\end{aligned}$$

Let $N \to \infty$ we obtain $\mathbb{E}\left[\left|X_t - X'_t\right|\right] = 0$. It means $X_t = X'_t$ almost surely for any $t \in [0; T]$. Since X and X' are càdlàg, they are indistinguishable on $[0, T]$. The proof is complete.

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Regular Articles

On the infinite time horizon approximation for Lévy-driven McKean-Vlasov SDEs with non-globally Lipschitz continuous and super-linearly growth drift and diffusion coefficients

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ABSTRACT

This paper studies the numerical approximation for McKean-Vlasov stochastic differential equations driven by Lévy processes. We propose a tamed-adaptive Euler-Maruyama scheme and consider its strong convergence in both finite and infinite time horizons when applying for some classes of Lévy-driven McKean-Vlasov stochastic differential equations with non-globally Lipschitz continuous and super-linearly growth drift and diffusion coefficients.

1. Introduction

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the d-dimensional process $X = (X_t)_{t\geq 0}$ solution to the following McKean-Vlasov stochastic differential equation (SDE) with jumps

$$dX_t = b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t, \mathcal{L}_{X_t})dW_t + c\left(X_{t-}, \mathcal{L}_{X_{t-}}\right) dZ_t, \tag{1}$$

for $t \geq 0$, where $X_0 = x_0 \in \mathbb{R}^d$ is a fixed initial value, $\mathcal{L}_{X_t}$ denotes the marginal law of the process X at time t, $W = (W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion, and $Z = (Z_t)_{t\geq 0}$ is a d-dimensional centered pure jump Lévy process whose Lévy measure ν satisfies $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(dz) < +\infty$. Two processes W and Z are supposed to be independent. The natural filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by W and Z. Let

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$\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$, and $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0^d)$ be the Borel σ-algebra on $\mathbb{R}_+ \times \mathbb{R}_0^d$. The Lévy-Itô decomposition of Z is given by

$$Z_t = \int_0^t \int_{\mathbb{R}_0^d} z(N(ds, dz) - \nu(dz)ds), \qquad \text{for any } t \geq 0,$$

where $N(dt, dz)$ is a Poisson random measure on the measurable space $(\mathbb{R}_+ \times \mathbb{R}_0^d, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0^d))$ associated with the intensity measure $\nu(dz)dt$. That is,

$$N([0,t] \times A) := \sum_{0 \leq s \leq t} \mathbf{1}_{\{\Delta Z_s \in A\}}, \text{ for any } t > 0, \text{ and } A \in \mathcal{B}(\mathbb{R}_0^d).$$

Here, the jump size of Z at instant s is defined as $\Delta Z_s := Z_s - Z_{s-} := Z_s - \lim_{u \uparrow s} Z_u$ for any $s > 0$, $\Delta Z_0 := 0$. The compensated Poisson random measure associated with $N(dt, dz)$ is denoted by $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of all probability measures defined on a measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and by

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}$$

the subset of probability measures with the finite second moment. As metric on the space $\mathcal{P}_2(\mathbb{R}^d)$, we use the $\mathcal{L}_2$-Wasserstein distance. That is, for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the $\mathcal{L}_2$-Wasserstein distance between μ and ν is defined as

$$\mathcal{W}_2(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \pi(dx, dy) \right)^{1/2},$$

where $\mathcal{C}(\mu, \nu)$ denotes all the couplings of μ and ν. That is, $\pi \in \mathcal{C}(\mu, \nu)$ if and only if $\pi(\cdot, \mathbb{R}^d) = \mu(\cdot)$ and $\pi(\mathbb{R}^d, \cdot) = \nu(\cdot)$.

The coefficients $b = (b_i)_{1 \leq i \leq d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma = (\sigma_{ij})_{1 \leq i,j \leq d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $c = (c_{ij})_{1 \leq i,j \leq d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable functions. The integral equation (1) now writes as

$$X_t = x_0 + \int_0^t b(X_s, \mathcal{L}_{X_s})ds + \int_0^t \sigma(X_s, \mathcal{L}_{X_s})dW_s + \int_0^t \int_{\mathbb{R}_0^d} c\left(X_{s-}, \mathcal{L}_{X_{s-}}\right) z\widetilde{N}(ds, dz), \quad t \geq 0.$$

The McKean-Vlasov process was first studied by McKean in [26] as a model for the Vlasov equation describing the time evolution of the distribution function of a plasma consisting of charged particles with long-range interaction. The process can be obtained as a limit of a mean-field system of interacting particles as the number of particles tends to infinity. The very first studies on the numerical approximation for McKean-Vlasov SDEs are the works of Ogawa [28], Kohatsu-Higa and Ogawa [18] and Bossy and Talay [3], where the authors considered the weak approximation of McKean-Vlasov SDEs with regular coefficients. However, the numerical approximation for McKean-Vlasov SDEs has only become active in the last decade.

Let $X_T^{(n)}$ be an approximation of X_T which depends on the values of W and Z at fixed equidistance times $t_k = \frac{kT}{n}, k = 0, 1, \ldots, n$. Then under some regularity conditions on the coefficients b, σ, and c, one may prove that

$$\|X_T^{(n)} - X_T\|_{L^p} := \left(\mathbb{E}[|X_T^{(n)} - X_T|^p]\right)^{1/p} \leq \frac{C(T)}{n^{\zeta_0}}, \tag{2}$$

for some positive constants p and ζ_0. In case the estimate (2) holds, we say that the $X_T^{(n)}$ converges at the rate of order ζ_0 in L^p-norm.

It is now well-known that for SDEs with super-linear growth coefficients, the Euler-Maruyama scheme may not converge in the L^p-norm (see [13]). Therefore, the numerical approximation for SDEs with super-linear growth coefficients has attracted lots of attention recently. New approximation schemes have been introduced to solve the problem, such as the tamed Euler-Maruyama scheme ([14,31,32,21]), the truncated Euler-Maruyama scheme ([24]), the implicit Euler-Maruyama scheme ([25]), the adaptive Euler-Maruyama scheme ([10]). In [7,19,20,22,23] the tamed Euler-Maruyama scheme has been developed to approximate McKean-Vlasov SDEs with super-linear growth coefficients. In [29], the authors introduced several adaptive Euler-Maruyama and Milstein schemes and studied their strong rates of convergence for McKean-Vlasov SDEs with super-linear drift. The paper [29] also discusses the numerical approximation for SDEs with super-linear diffusion coefficients, but it does not address the propagation of chaos for these SDEs. In [15], the authors introduced a multilevel Picard approximation, which has a low computational cost, for McKean-Vlasov SDEs with additive noise. In [6,7], the authors introduced the implicit Euler-Maruyama scheme and studied its convergence in L^p-norm for McKean-Vlasov SDEs with super-linearly growth drifts.

The McKean-Vlasov SDEs with jumps were studied by many authors with applications in many domains (see [12,8,9,1,11] and the reference therein). In [27], the authors considered McKean-Vlasov SDEs driven by infinite activity Lévy processes with super-linearly growth coefficients. They proved the existence and uniqueness of the solution and proposed a tamed Euler-Maruyama approximation for the associated interacting particle system and proved that the rate of its convergence in L^p-norm is arbitrarily close to 0.5.

In some applications, it is necessary to approximate X_T for T large, e.g., to approximate the invariant distribution of process X (see Section 3 in [10]). However, since the proof of convergence of many schemes often involves Gronwall's inequality, the quantity $C(T)$ may grow exponentially fast to T. Recently, there have been some attempts to introduce numerical schemes where $C(T)$ does not depend on T. The paper [10] proposed an adaptive Euler-Maruyama scheme for SDEs where each stepsize is adjusted to the size of the current values of X, and showed its strong convergence in the interval $[0, \infty)$ when applying for SDEs whose coefficients b and σ satisfy the contractive Lipschitz condition (Assumption 9 in [10]), b is polynomial growing Lipschitz continuous, and σ is bounded and globally Lipschitz continuous. The paper [16] introduced a tamed-adaptive Euler-Maruyama scheme and considered its rate of convergence in L^1-norm when applying for one-dimensional SDEs whose diffusion coefficient σ is super-linearly growing. In [17], the result of [16] was generalized for one-dimensional SDEs with jumps.

This paper aims to generalize the result of the papers [10,16,17] for multi-dimensional McKean-Vlasov SDEs with jumps. In particular, we propose a tamed-adaptive Euler-Maruyama approximation scheme for the Lévy-driven SDEs (1) where σ and b are non-globally Lipschitz continuous and super-linearly growing, and c is Lipschitz continuous. We will study the strong convergence of the scheme in both finite and infinite time intervals. In [29,27], the authors only considered the strong convergence of their adaptive scheme in a fixed time interval. To the best of our knowledge, our tamed-adaptive Euler-Maruyama scheme is the first approximation method for Lévy-driven McKean-Vlasov SDEs that could be shown to converge in an infinite time horizon.

The most challenging part of this work is to show the integrability of the exact and approximate solutions up to a certain order. One usually uses a stopping time technique to show the integrability. However, it was pointed out in [7] that one has to be very careful with this technique. Our framework is very close to the one in the preprint [27]. Unfortunately, the proof for the finiteness of the p-moment of the solution (Theorem 2.1 in [27]) is incomplete. In this paper, we address this issue by employing an alternative approach. In particular, we will use the method of induction for the order of moments. Thanks to the binomial theorem, the bound of higher-order moments will be obtained from the bounds of lower-order moments. Another challenging part is to show that the adaptive scheme is well-defined. We have to modify the approach in the work of Fang and Giles [10]. This new approach allows us to write the approximate solution in the form of an Itô process, thus the Itô formula can be applied to the calculation.

We denote the vector Euclidean norm of v by $|v| := (|v_1|^2 + |v_2|^2 + \ldots + |v_d|^2)^{\frac{1}{2}}$, the inner product of vectors v and w by $\langle v, w\rangle := v_1w_1 + v_2w_2 + \ldots + v_dw_d$ for any $v, w \in \mathbb{R}^d$, the Frobenius matrix norm by $|A| := \sqrt{\sum_{i,j=1}^d A_{ij}^2}$ for all $A \in \mathbb{R}^{d\times d}$, the integer part of a by $[a]$, and the transpose of a vector or matrix A by A^{T}. Moreover, the binomial coefficients are denoted by $\binom{n}{k} := \frac{n!}{k!(n-k)!}$. In all that follows, we denote by C positive constants whose value may change from one line to the next.

The rest of this paper is structured as follows. In Section 2, we introduce assumptions on the coefficients of equation (1) and show some moment estimates under these assumptions. In Section 3, we prove the propagation of chaos. In Section 4, we introduce the tamed-adaptive Euler-Maruyama scheme and prove that it is well-defined and converges in L^2-norm. Section 5 presents a numerical study for the tamed-adaptive Euler-Maruyama scheme.

2. Model assumptions and moment estimates

Throughout this paper, we always assume that the following assumptions hold.

A1. There exists a positive constant L such that for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$2\langle x, b(x,\mu)\rangle + |\sigma(x,\mu)|^2 + |c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) \le L\left(1 + |x|^2 + \mathcal{W}_2^2(\mu,\delta_0)\right).$$

A2. There exist constants $\kappa_1 \ge 1, \kappa_2 \ge 1$, $L_1 \in \mathbb{R}$ and $L_2 \ge 0$ such that for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}&2\langle x - \overline{x}, b(x,\mu) - b(\overline{x},\overline{\mu})\rangle + \kappa_1|\sigma(x,\mu) - \sigma(\overline{x},\overline{\mu})|^2 + \kappa_2|c(x,\mu) - c(\overline{x},\overline{\mu})|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz)\\ &\le L_1|x-\overline{x}|^2 + L_2\mathcal{W}_2^2(\mu,\overline{\mu}).\end{aligned}$$

A3. There exist constants $L > 0$ and $\ell \ge 0$ such that for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|b(x,\mu) - b(\overline{x},\overline{\mu})| \le L\left(1 + |x|^\ell + |\overline{x}|^\ell\right)\left(|x-\overline{x}| + \mathcal{W}_2(\mu,\overline{\mu})\right).$$

A4. There exists an even integer $p_0 \in [2, +\infty)$ such that $\int_{|z|>1} |z|^{2p_0}\nu(dz) < \infty$ and $\int_{0<|z|\le 1} |z|\nu(dz) < \infty$.

A5. There exists a positive constant L_3 such that for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|c(x,\mu)| \le L_3\left(1 + |x| + \mathcal{W}_2(\mu,\delta_0)\right).$$

A6. There exist constants $\gamma_1 \in \mathbb{R}$, $\gamma_2 \ge 0$ and $\eta \ge 0$ such that for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}&\langle x, b(x,\mu)\rangle + \frac{p_0-1}{2}|\sigma(x,\mu)|^2 + \frac{1}{2L_3}|c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|\left((1+L_3|z|)^{p_0-1} - 1\right)\nu(dz)\\ &\le \gamma_1|x|^2 + \gamma_2\mathcal{W}_2^2(\mu,\delta_0) + \eta,\end{aligned}$$

where the constants p_0 and L_3 are defined in **A4** and **A5**, respectively.

Remark 2.1. (i) It follows from Condition **A3** that, for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|b(x,\mu)| \leq |b(0,\delta_0)| + L\left(1+|x|^\ell\right)\left(|x| + \mathcal{W}_2(\mu,\delta_0)\right).$$

(ii) Assume that Condition **A2** holds for $\kappa_1 = \kappa_2 = 1+\varepsilon$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$ with a constant $\varepsilon > 0$. This, combined with Condition **A3** and Cauchy's inequality, implies that

$$\begin{aligned}&(1+\varepsilon)\left|\sigma(x,\mu)-\sigma(\overline{x},\overline{\mu})\right|^2 + (1+\varepsilon)\left|c(x,\mu)-c(\overline{x},\overline{\mu})\right|^2 \int_{\mathbb{R}_0^d} |z|^2 \nu(dz)\\ &\leq L\widetilde{L}\left(1+|x|^\ell+|\overline{x}|^\ell\right)\left(|x-\overline{x}|^2 + \mathcal{W}_2^2(\mu,\overline{\mu})\right),\end{aligned}$$

with $\widetilde{L} = \max\{3, L_1, L_2\}$ for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$. This yields that, for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\left|\sigma(x,\mu)-\sigma(\overline{x},\overline{\mu})\right|^2 + \left|c(x,\mu)-c(\overline{x},\overline{\mu})\right|^2 \int_{\mathbb{R}_0^d} |z|^2 \nu(dz) \leq \frac{L\widetilde{L}}{1+\varepsilon}\left(1+|x|^\ell+|\overline{x}|^\ell\right)\left(|x-\overline{x}|^2 + \mathcal{W}_2^2(\mu,\overline{\mu})\right).$$

(iii) From (ii), we have that for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}|\sigma(x,\mu)|^2 &\leq 2\left|\sigma(x,\mu)-\sigma(0,\delta_0)\right|^2 + 2\left|\sigma(0,\delta_0)\right|^2\\ &\leq 2\frac{L\widetilde{L}}{1+\varepsilon}\left(1+|x|^\ell\right)\left(|x|^2 + \mathcal{W}_2^2(\mu,\delta_0)\right) + 2\left|\sigma(0,\delta_0)\right|^2,\end{aligned}$$

and similarly,

$$|c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) \leq 2\frac{L\widetilde{L}}{1+\varepsilon}\left(1+|x|^\ell\right)\left(|x|^2 + \mathcal{W}_2^2(\mu,\delta_0)\right) + 2\left|c(0,\delta_0)\right|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz).$$

Remark 2.2. It follows from Condition **A6** that, for any $p \in [2, p_0]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\langle x, b(x,\mu)\rangle + \frac{p-1}{2}|\sigma(x,\mu)|^2 + \frac{1}{2L_3}|c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|\left(\left(1+L_3|z|\right)^{p-1} - 1\right)\nu(dz) \leq \gamma_1|x|^2 + \gamma_2\mathcal{W}_2^2(\mu,\delta_0) + \eta.$$

Proposition 2.3. *There exists a unique càdlàg process $X = (X_t)_{t\geq 0}$ satisfying the McKean-Vlasov SDE with jumps (1). Moreover, for any $p \in [2, p_0]$, there exists a positive constant C_p such that for any $t \geq 0$,*

$$\mathbb{E}\left[|X_t|^p\right] \leq \begin{cases} C_p(1+e^{\gamma p t}) & \text{if} \quad \gamma \neq 0,\\ C_p(1+t)^{p/2} & \text{if} \quad \gamma = 0, p = 2 \quad \text{or} \quad \gamma = 0, \gamma_2 > 0, p \in (2, p_0],\\ C_p(1+t)^p & \text{if} \quad \gamma = 0, \gamma_2 = 0, p \in (2, p_0],\end{cases} \tag{3}$$

where $\gamma = \gamma_1 + \gamma_2$.

Note that if $\gamma < 0$, we have that $\sup_{t\geq 0} \mathbb{E}\left[|X_t|^p\right] \leq 2C_p$.

Proof. Step 1: It follows from [27, Theorem 2.1] that there exists a unique càdlàg process $X = (X_t)_{t\geq 0}$ satisfying the McKean-Vlasov SDE with jumps (1), and $\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^2\right] \leq K$, where $T > 0$ is a fixed constant and $K := K(|x_0|^2, d, L, L_1, T)$ is a positive constant.

Step 2: We show that for any even integer $p \in [2, p_0]$ and $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^p\right] \le C(T,p). \tag{4}$$

Note that (4) holds for $p = 2$ due to Step 1. Next, we assume that (4) is valid for any even integer $q \in [2, p-2]$. That is,

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^q\right] \le C(T,q). \tag{5}$$

Now, for $\lambda \in \mathbb{R}$ and even integer $p \in [2, p_0]$, applying Itô's formula to $e^{-\lambda t}|X_t|^p$, we obtain that

$$\begin{aligned}
e^{-\lambda t}|X_t|^p &= |x_0|^p + \int_0^t e^{-\lambda s}\Big[-\lambda |X_s|^p + p|X_s|^{p-2}\left\langle X_s, b\left(X_s,\mathcal{L}_{X_s}\right)\right\rangle + \frac{p}{2}|X_s|^{p-2}\left|\sigma\left(X_s,\mathcal{L}_{X_s}\right)\right|^2 \\
&\quad + \frac{p(p-2)}{2}|X_s|^{p-4}\left|X_s^{\top}\sigma\left(X_s,\mathcal{L}_{X_s}\right)\right|^2\Big]ds + p\int_0^t e^{-\lambda s}|X_s|^{p-2}\left\langle X_s, \sigma\left(X_s,\mathcal{L}_{X_s}\right)dW_s\right\rangle \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} e^{-\lambda s}\left(\left|X_{s-} + c\left(X_{s-},\mathcal{L}_{X_{s-}}\right)z\right|^p - |X_{s-}|^p\right)\widetilde{N}(ds,dz) \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} e^{-\lambda s}\Big(\left|X_s + c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^p - |X_s|^p - p|X_s|^{p-2}\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\Big)\nu(dz)ds.
\end{aligned} \tag{6}$$

Using the binomial theorem, we expand the last integral in (6) as follows

$$\begin{aligned}
&\left|X_s + c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^p = \left(\left|X_s + c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^2\right)^{p/2} \\
&= |X_s|^p + p|X_s|^{p-2}\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle + \frac{p}{2}|X_s|^{p-2}\left|c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^2 \\
&\quad + \sum_{i=2}^{p/2}\binom{p/2}{i}|X_s|^{p-2i}\left(\left|c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^2 + 2\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\right)^i.
\end{aligned} \tag{7}$$

Next, using the binomial theorem and the inequality $\langle x, y\rangle \le |x||y|$, we have

$$\begin{aligned}
&|X_s|^{p-2i}\left(\left|c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^2 + 2\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\right)^i \\
&\le \left|c\left(X_s,\mathcal{L}_{X_s}\right)\right|^2\sum_{j=0}^{i}\binom{i}{j}2^j|X_s|^{p-2i+j}\left|c\left(X_s,\mathcal{L}_{X_s}\right)\right|^{2i-j-2}|z|^{2i-j}.
\end{aligned}$$

Note that $\left|c\left(X_s,\mathcal{L}_{X_s}\right)\right| \le L_3\left(1 + |X_s| + \mathcal{W}_2(\mathcal{L}_{X_s},\delta_0)\right) \le L_3\left(1 + |X_s| + \sqrt{\mathbb{E}[|X_s|^2]}\right)$ due to **A5** and the equality $\mathcal{W}_2^2\left(\mathcal{L}_{X_s},\delta_0\right) = \mathbb{E}[|X_s|^2]$. Using the binomial theorem, we get

$$\begin{aligned}
&|X_s|^{p-2i}\left(\left|c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^2 + 2\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\right)^i \\
&\le \left|c\left(X_s,\mathcal{L}_{X_s}\right)\right|^2\sum_{j=0}^{i}\binom{i}{j}2^j|z|^{2i-j}L_3^{2i-j-2}\Bigg(|X_s|^{p-2} + (2i-j-2)\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)|X_s|^{p-3}
\end{aligned}$$

$$
+ \sum_{k=2}^{2i-j-2} \binom{2i-j-2}{k} \left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k |X_s|^{p-2-k} \Bigg).
$$

Applying the Young's inequality to $|x|^{(p-2)/2}$ and $y|x|^{(p-4)/2}$, we have $y|x|^{p-3} \le \frac{1}{2}(|x|^{p-2}+y^2|x|^{p-4})$ valid for any $x \in \mathbb{R}^d, y > 0$. This, combined with the equalities $\sum_{j=0}^{i} \binom{i}{j} a^j = (1+a)^i$ and $\sum_{j=0}^{i} \binom{i}{j} j a^j = ia(1+a)^{i-1}$ valid for any $a \in \mathbb{R}$, and **A5**, deduces

$$
\begin{aligned}
&|X_s|^{p-2i} \left(|c(X_s, \mathcal{L}_{X_s}) z|^2 + 2\langle X_s, c(X_s, \mathcal{L}_{X_s}) z\rangle\right)^i \\
&\le |c(X_s, \mathcal{L}_{X_s})|^2 |X_s|^{p-2} \sum_{j=0}^{i} \binom{i}{j} 2^j |z|^{2i-j} L_3^{2i-j-2} \left(i - \frac{j}{2}\right) + |c(X_s, \mathcal{L}_{X_s})|^2 \sum_{j=0}^{i} \binom{i}{j} 2^j |z|^{2i-j} L_3^{2i-j-2} \\
&\quad \times \left(\left(i - \frac{j}{2} - 1\right) \left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^2 |X_s|^{p-4} + \sum_{k=2}^{2i-j-2} \binom{2i-j-2}{k} \left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k |X_s|^{p-2-k} \right) \\
&= |c(X_s, \mathcal{L}_{X_s})|^2 |X_s|^{p-2} \left(\frac{i}{L_3^2} \left(L_3^2|z|^2 + 2L_3|z|\right)^i - \frac{|z|}{L_3} i \left(L_3^2|z|^2 + 2L_3|z|\right)^{i-1} \right) \\
&\quad + |c(X_s, \mathcal{L}_{X_s})|^2 \sum_{j=0}^{i} \binom{i}{j} 2^j |z|^{2i-j} \\
&\quad \times L_3^{2i-j-2} \left(\left(i - \frac{j}{2} - 1\right) \left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^2 |X_s|^{p-4} + \sum_{k=2}^{2i-j-2} \binom{2i-j-2}{k} \left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k |X_s|^{p-2-k} \right) \\
&\le |c(X_s, \mathcal{L}_{X_s})|^2 |X_s|^{p-2} \frac{i|z|}{L_3} (1 + L_3|z|) \left(L_3^2|z|^2 + 2L_3|z|\right)^{i-1} \\
&\quad + \sum_{j=0}^{i} |z|^{2i-j} Q_p\left(p-2, 2i-j, |X_s|, 1+\sqrt{\mathbb{E}[|X_s|^2]}\right),
\end{aligned}
$$

where $2i - j \ge 2$ and $Q_q(m,n,x,y)$ is a certain polynomial of the form $Q_q(m,n,x,y) = \sum_{\ell_1 \le m, \ell_2 \le n, \ell_1+\ell_2=q} c_{\ell_1 \ell_2} x^{\ell_1} y^{\ell_2}$. This, together with the fact that $\sum_{i=2}^{p/2} \binom{p/2}{i} i a^{i-1} = \frac{p}{2}((1+a)^{p/2-1} - 1)$ valid for any $a \in \mathbb{R}$, yields to

$$
\begin{aligned}
&\sum_{i=2}^{p/2} \binom{p/2}{i} |X_s|^{p-2i} \left(|c(X_s, \mathcal{L}_{X_s}) z|^2 + 2\langle X_s, c(X_s, \mathcal{L}_{X_s}) z\rangle\right)^i \\
&\le \frac{p}{2} |X_s|^{p-2} |c(X_s, \mathcal{L}_{X_s})|^2 \frac{|z|}{L_3} \left((1+L_3|z|)^{p-1} - L_3|z| - 1\right) \\
&\quad + \sum_{i=2}^{p/2} \sum_{j=0}^{i} \binom{p/2}{i} |z|^{2i-j} Q_p\left(p-2, 2i-j, |X_s|, 1+\sqrt{\mathbb{E}[|X_s|^2]}\right).
\end{aligned} \tag{8}
$$

As a consequence of (7) and (8), we have shown that for any $s \ge 0$,

$$
\begin{aligned}
&|X_s + c(X_s, \mathcal{L}_{X_s}) z|^p - |X_s|^p - p|X_s|^{p-2} \langle X_s, c(X_s, \mathcal{L}_{X_s}) z\rangle \\
&\le \frac{p}{2L_3} |X_s|^{p-2} |c(X_s, \mathcal{L}_{X_s})|^2 |z| \left((1+L_3|z|)^{p-1} - 1\right) \\
&\quad + \sum_{i=2}^{p/2} \sum_{j=0}^{i} \binom{p/2}{i} |z|^{2i-j} Q_p\left(p-2, 2i-j, |X_s|, 1+\sqrt{\mathbb{E}[|X_s|^2]}\right).
\end{aligned} \tag{9}
$$

Therefore, substituting (9) into (6), we get that for any $t \geq 0$,

$$
\begin{aligned}
e^{-\lambda t}|X_t|^p \leq |x_0|^p &+ p\int_0^t e^{-\lambda s}|X_s|^{p-2}\Big[-\frac{\lambda}{p}|X_s|^2 + \langle X_s, b(X_s, \mathcal{L}_{X_s})\rangle + \frac{p-1}{2}|\sigma(X_s, \mathcal{L}_{X_s})|^2 \\
&+ \frac{1}{2L_3}|c(X_s, \mathcal{L}_{X_s})|^2 \int_{\mathbb{R}_0^d} |z|\left((1+L_3|z|)^{p-1}-1\right)\nu(dz)\Big]ds \\
&+ \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\sum_{i=2}^{p/2}\sum_{j=0}^{i}\binom{p/2}{i}|z|^{2i-j}Q_p\left(p-2, 2i-j, |X_s|, 1+\sqrt{\mathbb{E}[|X_s|^2]}\right)\nu(dz)ds \\
&+ p\int_0^t e^{-\lambda s}|X_s|^{p-2}\langle X_s, \sigma(X_s, \mathcal{L}_{X_s})dW_s\rangle + \int_0^t\int_{\mathbb{R}_0^d} e^{-\lambda s}\left(\left|X_{s-} + c\left(X_{s-}, \mathcal{L}_{X_{s-}}\right)z\right|^p - |X_{s-}|^p\right)\widetilde{N}(ds, dz).
\end{aligned}
$$

Now, we define $\tau_N := \inf\{t \geq 0 : |X_t| \geq N\}$ for each $N > 0$. Choosing $\lambda = \gamma_1 p$ and using Condition **A6**, Remark 2.2 together with $\mathcal{W}_2^2(\mathcal{L}_{X_s}, \delta_0) = \mathbb{E}[|X_s|^2]$, we obtain that

$$
\begin{aligned}
\mathbb{E}\left[e^{-\gamma_1 p(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] &\leq |x_0|^p + p\int_0^t e^{-\gamma_1 ps}\mathbb{E}\left[|X_s|^{p-2}\right]\left(\gamma_2\mathcal{W}_2^2(\mathcal{L}_{X_s}, \delta_0) + \eta\right)ds \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} e^{-\gamma_1 ps}\sum_{i=2}^{p/2}\sum_{j=0}^{i}\binom{p/2}{i}|z|^{2i-j}\mathbb{E}\left[Q_p\left(p-2, 2i-j, |X_s|, 1+\sqrt{\mathbb{E}[|X_s|^2]}\right)\right]\nu(dz)ds \\
&\leq |x_0|^p + p\int_0^t e^{-\gamma_1 ps}\mathbb{E}\left[|X_s|^{p-2}\right]\left(\gamma_2\mathbb{E}\left[|X_s|^2\right] + \eta\right)ds \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} e^{-\gamma_1 ps}\sum_{i=2}^{p/2}\sum_{j=0}^{i}\binom{p/2}{i}|z|^{2i-j}\mathbb{E}\left[Q_p\left(p-2, 2i-j, |X_s|, 1+\sqrt{\mathbb{E}[|X_s|^2]}\right)\right]\nu(dz)ds. \qquad (10)
\end{aligned}
$$

Next, using $e^{-\gamma_1 p(t\wedge\tau_N)} \geq e^{-|\gamma_1|pT}$ for any $t \in [0;T]$, and the induction assumption (5), there exists a positive constant $C(T,p)$ which does not depend on N such that

$$
\sup_{t\in[0,T]} \mathbb{E}\left[|X_{t\wedge\tau_N}|^p\right] \leq C(T,p). \qquad (11)
$$

This yields to $\sup_{t\in[0,T]}\mathbb{P}(\tau_N < t) \leq \frac{C(T,p)}{N^p}$. This fact, together with the monotonicity of τ_N with respect to N, implies that τ_N tends to infinity a.s. as N tends to infinity. Now, it suffices to let $N \uparrow \infty$ and use Fatou's lemma for the left-hand side of (11) to get that $\sup_{t\in[0,T]}\mathbb{E}[|X_t|^p] \leq C(T,p)$. Thus, by the induction principle, we have shown (4).

Step 3: We show (3) for any even integer $p \in [2, p_0]$.

First, applying (6) to $\lambda = 2\gamma$ and $p = 2$, and using **A6**, we get

$$
e^{-2\gamma t}|X_t|^2 \leq |x_0|^2 + 2\int_0^t e^{-2\gamma s}\left(-\gamma_2|X_s|^2 + \gamma_2\mathcal{W}_2^2(\mathcal{L}_{X_s}, \delta_0) + \eta\right)ds + 2\int_0^t e^{-2\gamma s}\langle X_s, \sigma(X_s, \mathcal{L}_{X_s})dW_s\rangle
$$

$$+\int_0^t\int_{\mathbb{R}_0^d} e^{-2\gamma s}\left(\left|X_{s-}+c\left(X_{s-},\mathcal{L}_{X_{s-}}\right)z\right|^2-|X_{s-}|^2\right)\widetilde{N}(ds,dz). \tag{12}$$

Thanks to the fact that $\mathcal{W}_2^2\left(\mathcal{L}_{X_s},\delta_0\right)=\mathbb{E}[|X_s|^2]$, and the estimate (4), we get $\mathbb{E}\left[e^{-2\gamma t}|X_t|^2\right]\le|x_0|^2+2\eta\int_0^t e^{-2\gamma s}ds$, which yields to

$$\mathbb{E}\left[|X_t|^2\right]\le\begin{cases}\left(|x_0|^2+\dfrac{\eta}{\gamma}\right)e^{2\gamma t}-\dfrac{\eta}{\gamma} & \text{if}\quad \gamma\neq 0,\\ |x_0|^2+2\eta t & \text{if}\quad \gamma=0.\end{cases}$$

Thus, (3) holds for $p=2$.

Now, we suppose that (3) is valid for all even integer $q\in[2,p-2]$. That is,

$$\mathbb{E}\left[|X_t|^q\right]\le\begin{cases}C_q(1+e^{\gamma qt}) & \text{if}\quad \gamma\neq 0,\\ C_q(1+t)^{q/2} & \text{if } \gamma=0, q=2 \ \text{ or } \ \gamma=0,\gamma_2>0, q\in(2,p-2],\\ C_q(1+t)^q & \text{if}\quad \gamma=0,\gamma_2=0,q\in(2,p-2].\end{cases} \tag{13}$$

We are going to show that (3) holds for even integer p. For this, it suffices to use (10), the inductive assumption (13) and Condition **A4**.

Case $\gamma\neq 0$: We have

$$\mathbb{E}\left[e^{-\gamma_1 p(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right]\le|x_0|^p+C\int_0^t e^{-\gamma_1 ps}\left(1+e^{\gamma pt}\right)ds=|x_0|^p+C\int_0^t\left(e^{-\gamma_1 ps}+e^{\gamma_2 ps}\right)ds.$$

Thanks to fact that $\tau_N\uparrow\infty$ a.s. as $N\uparrow\infty$, applying Fatou's lemma, we get

$$\mathbb{E}\left[e^{-\gamma_1 pt}|X_t|^p\right]\le|x_0|^p+C\int_0^t\left(e^{-\gamma_1 ps}+e^{\gamma_2 ps}\right)ds.$$

This implies that $\mathbb{E}\left[|X_t|^p\right]\le C\left(1+e^{\gamma pt}\right)$.

Case $\gamma=0$: When $\gamma_2>0$, we have

$$\mathbb{E}\left[e^{-\gamma_1 p(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right]\le|x_0|^p+C\int_0^t e^{-\gamma_1 ps}\left(1+s\right)^{p/2}ds\le|x_0|^p+C\left(1+t\right)^{p/2}\int_0^t e^{-\gamma_1 ps}ds.$$

Then, letting $N\uparrow\infty$ and using Fatou's Lemma and the fact that $-\gamma_1=\gamma_2$, we obtain

$$\mathbb{E}\left[e^{\gamma_2 pt}|X_t|^p\right]\le|x_0|^p+C\left(1+t\right)^{p/2}\int_0^t e^{\gamma_2 ps}ds\le|x_0|^p+\frac{C}{\gamma_2 p}\left(1+t\right)^{p/2}e^{\gamma_2 pt}.$$

Hence, $\mathbb{E}\left[|X_t|^p\right]\le C\left(1+t\right)^{p/2}$.

When $\gamma_2=\gamma_1=0$, we have $\mathbb{E}\left[|X_{t\wedge\tau_N}|^p\right]\le|x_0|^p+C\int_0^t\left(1+s\right)^{p-1}ds\le C\left(1+t\right)^p$. Then, letting $N\uparrow\infty$, we obtain $\mathbb{E}\left[|X_t|^p\right]\le C\left(1+t\right)^p$.

Consequently, (3) holds for even integer p. Due to the induction principle, (3) is valid for any even integer $p\in[2,p_0]$. Finally, (3) is also valid for any $p\in[2,p_0]$ thanks to Hölder's inequality. This finishes the proof. □

3. Propagation of chaos

For $N \in \mathbb{N}$, suppose that (W^i, Z^i) are independent copies of the couple (W, Z) for $i \in \{1, \dots, N\}$. Let $N^i(dt, dz)$ be the Poisson random measure associated with the jumps of the Lévy process Z^i with intensity measure $\nu(dz)dt$, and $\widetilde{N}^i(dt, dz) := N^i(dt, dz) - \nu(dz)dt$ be the compensated Poisson random measure associated with $N^i(dt, dz)$. Thus, the Lévy-Itô decomposition of Z^i is given by $Z^i_t = \int_0^t \int_{\mathbb{R}^d_0} z\widetilde{N}^i(ds, dz)$ for $t \geq 0$. We now consider the system of non-interacting particles which is associated with the Lévy-driven McKean-Vlasov SDE (1), where the state $X^i = (X^i_t)_{t\geq 0}$ of the particle i is defined by

$$X^i_t = x_0 + \int_0^t b(X^i_s, \mathcal{L}_{X^i_s})ds + \int_0^t \sigma(X^i_s, \mathcal{L}_{X^i_s})dW^i_s + \int_0^t \int_{\mathbb{R}^d_0} c\left(X^i_{s-}, \mathcal{L}_{X^i_{s-}}\right) z\widetilde{N}^i(ds, dz),$$

for any $t \geq 0$ and $i \in \{1, \dots, N\}$.

For $\boldsymbol{x}^N := (x_1, x_2, \dots, x_N), \boldsymbol{y}^N := (y_1, y_2, \dots, y_N) \in \mathbb{R}^{dN}$, we have $\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \delta_0) = \frac{1}{N}\sum_{i=1}^N |x_i|^2$. Here, the empirical measure is defined by $\mu^{\boldsymbol{x}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}(dx)$, where δ_x denotes the Dirac measure at x. Moreover, a standard bound for the Wasserstein distance between two empirical measures $\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}$ is given by

$$\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}) \leq \frac{1}{N}\sum_{i=1}^N |x_i - y_i|^2 = \frac{1}{N}\left|\boldsymbol{x}^N - \boldsymbol{y}^N\right|^2,$$

(see (1.24) of [4]).

Now, the measure $\mathcal{L}_{X_t}$ at each time t is approximated by the empirical measure $\mu_t^{\boldsymbol{X}^N}(dx) := \dfrac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}_t}(dx)$, where $\boldsymbol{X}^N = (\boldsymbol{X}^N_t)_{t\geq 0} = (X^{1,N}_t, \dots, X^{N,N}_t)^{\mathsf{T}}_{t\geq 0}$, which is called the system of interacting particles, is the solution to the $\mathbb{R}^{dN}$-dimensional Lévy-driven SDE with components $X^{i,N} = (X^{i,N}_t)_{t\geq 0}$

$$X^{i,N}_t = x_0 + \int_0^t b(X^{i,N}_s, \mu_s^{\boldsymbol{X}^N})ds + \int_0^t \sigma(X^{i,N}_s, \mu_s^{\boldsymbol{X}^N})dW^i_s + \int_0^t \int_{\mathbb{R}^d_0} c\left(X^{i,N}_{s-}, \mu_{s-}^{\boldsymbol{X}^N}\right) z\widetilde{N}^i(ds, dz), \tag{14}$$

for any $t \geq 0$ and $i \in \{1, \dots, N\}$.

Observe that the interacting particle system $\boldsymbol{X}^N = (X^{i,N})^{\mathsf{T}}_{i\in\{1,\dots,N\}}$ can be viewed as an ordinary Lévy-driven SDE with random coefficients taking values in $\mathbb{R}^{dN}$. Therefore, under Conditions **A1**, **A2** and **A3**, by applying Theorem 2.1 in [27], there exists a unique càdlàg solution to (14) such that $\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[|X^{i,N}_t|^2\right] \leq K$, for any $N \in \mathbb{N}$, where $K > 0$ does not depend on N.

Proposition 3.1. *For any $p \in [2, p_0]$, there exists a positive constant C_p such that for any $t \geq 0$,*

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[|X^{i,N}_t|^p\right] \leq \begin{cases} C_p(1+e^{\gamma p t}) & \text{if } \gamma \neq 0, \\ C_p(1+t)^{p/2} & \text{if } \gamma = 0, p = 2 \text{ or } \gamma = 0, \gamma_2 > 0, p \in (2, p_0], \\ C_p(1+t)^p & \text{if } \gamma = 0, \gamma_2 = 0, p \in (2, p_0], \end{cases}$$

where $\gamma = \gamma_1 + \gamma_2$.

Note that when $\gamma < 0$, we have that $\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[|X^{i,N}_t|^p\right] \leq 2C_p$.

Proof. The proof follows the same lines as the one of Proposition 2.3, thus we omit it. □

The propagation of chaos for SDEs with super-linear growth drift and global Lipschitz diffusion coefficients has been studied recently (see [7,19,20] for SDEs without jumps, and [8,9,27] for SDEs with jumps). In the following, we present a propagation of chaos result under more relaxed conditions than those existing in the literature. Specifically, we achieve the same type of propagation of chaos as in [7,19,20] for a class of SDEs with jumps, whose drift and diffusion can exhibit super-linear growth. An additional novel aspect of our approach is that it enables us to establish a bound that is independent of time (see (16)). To simplify the exposition, we define

$$\varphi(N) = N^{-1/2}\mathbf{1}_{\{d<4\}} + N^{-1/2}\ln N\mathbf{1}_{\{d=4\}} + N^{-2/d}\mathbf{1}_{\{d>4\}}.$$

Proposition 3.2. *There exists a positive constant C_T, which does not depend on N, such that*

$$\max_{i\in\{1,\dots,N\}}\sup_{t\in[0,T]}\mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] \le C_T\varphi(N). \tag{15}$$

Assume further that $L_1 + L_2 < 0$ and $\gamma = \gamma_1 + \gamma_2 < 0$. Then, there exists a positive constant C, which depends neither on N nor on T, such that

$$\max_{i\in\{1,\dots,N\}}\sup_{t\ge 0}\mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] \le C\varphi(N). \tag{16}$$

Proof. Applying Itô's formula and Condition **A2**, we obtain that for any $t \ge 0$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}
e^{-\lambda t}|X_t^i - X_t^{i,N}|^2 &\le \int_0^t e^{-\lambda s}\Big(-\lambda|X_s^i - X_s^{i,N}|^2 + 2\langle X_s^i - X_s^{i,N}, b(X_s^i, \mathcal{L}_{X_s^i}) - b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})\rangle \\
&\quad + |\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})|^2 + |c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}|^2 \int_{\mathbb{R}_0^d}|z|^2\nu(dz)\Big)ds + \mathcal{M}_t \\
&\le \int_0^t e^{-\lambda s}\Big(-\lambda|X_s^i - X_s^{i,N}|^2 + L_1|X_s^i - X_s^{i,N}|^2 + L_2\mathcal{W}_2^2(\mathcal{L}_{X_s^i}, \mu_s^{\boldsymbol{X}^N})\Big)ds + \mathcal{M}_t,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_t &= 2\int_0^t e^{-\lambda s}\langle X_s^i - X_s^{i,N}, (\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}))dW_s^i\rangle \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} e^{-\lambda s}\left(\left|X_{s-}^i - X_{s-}^{i,N} + \left(c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N})\right)z\right|^2 - |X_{s-}^i - X_{s-}^{i,N}|^2\right)\widetilde{N}^i(ds,dz).
\end{aligned}$$

Therefore, taking the expectation and using the estimate $\mathcal{W}_2^2(\mathcal{L}_{X_s^i}, \mathcal{L}_{X_s^{i,N}}) \le \mathbb{E}[|X_s^i - X_s^{i,N}|^2]$, we obtain that for any $\varepsilon > 0$,

$$\begin{aligned}
e^{-\lambda t}\mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] &\le \int_0^t e^{-\lambda s}\left((-\lambda + L_1)\mathbb{E}\left[|X_s^i - X_s^{i,N}|^2\right] + L_2\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X_s^i}, \mu_s^{\boldsymbol{X}^N})\right]\right)ds \\
&\le \int_0^t e^{-\lambda s}\Bigg((-\lambda + L_1)\mathbb{E}\left[|X_s^i - X_s^{i,N}|^2\right]
\end{aligned}$$

$$+ L_2\Bigg(\big(1+\frac{\varepsilon}{L_2}\big)\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X_s^i}, \mathcal{L}_{X_s^{i,N}})\right] + \big(1+\frac{L_2}{\varepsilon}\big)\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X_s^{i,N}}, \mu_s^{\boldsymbol{X}^N})\right] \Bigg)\Bigg) ds$$

$$\leq \int_0^t e^{-\lambda s}\left((-\lambda + L_1 + L_2 + \varepsilon)\,\mathbb{E}\left[|X_s^i - X_s^{i,N}|^2\right] + L_2\big(1+\frac{L_2}{\varepsilon}\big)\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X_s^{i,N}}, \mu_s^{\boldsymbol{X}^N})\right]\right) ds. \tag{17}$$

Moreover, from Proposition 3.1, we have that for any $p \in (4, p_0]$, $\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^{i,N}\right|^p\right] \leq C_T$, for some constant $C_T > 0$. This, together with [5, Theorem 5.8], deduces that

$$\max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X_s^{i,N}}, \mu_s^{\boldsymbol{X}^N})\right] \leq C\varphi(N), \tag{18}$$

for any $s \in [0, T]$, where the positive constant C does not depend on the time.

Consequently, it suffices to choose $\lambda = L_1 + L_2 + \varepsilon$ in (17) and use the estimate (18) to conclude (15).

Finally, when $\gamma < 0$, it follows from Proposition 3.1 that for any $p \in (4, p_0]$,

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^{i,N}\right|^q\right] \leq C,$$

where the positive constant C does not depend on T. Furthermore, when $L_1 + L_2 < 0$, one can always choose ε sufficiently small such that $\lambda < 0$. This allows us to conclude (16). The desired proof follows. □

4. Tamed-adaptive Euler-Maruyama scheme

4.1. Definition

For each $\Delta \in (0, 1)$, we define tamed versions of σ and c by

$$\sigma_\Delta(x,\mu) = \frac{\sigma(x,\mu)}{1+\Delta^{1/2}|\sigma(x,\mu)|(1+|x|)}, \quad c_\Delta(x,\mu) = \frac{c(x,\mu)}{1+\Delta^{1/2}|c(x,\mu)|(1+|x|+|b(x,\mu)|)}. \tag{19}$$

The time-step will be controlled by the function

$$h(x,\mu) = \frac{h_0}{(1+|b(x,\mu)|+|\sigma(x,\mu)|+|x|^\ell)^2 + |c(x,\mu)|^{p_0}}, \tag{20}$$

with $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and some positive constant h_0. Here, the constants ℓ and p_0 are respectively defined in Conditions **A3** and **A4**. The parameter h_0 is useful in practice since we can adjust it to control the step-size. In all that follows, to simplify the exposition, we take $h_0 = 1$ in the proofs.

Now, the tamed-adaptive Euler-Maruyama approximation of the solution to (14) is defined as follows: for all $i \in \{1, \ldots, N\}$, and $k \in \mathbb{N}$,

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0^{i,N} = x_0, \quad \widehat{\boldsymbol{X}}_{t_k}^N = \left(\widehat{X}_{t_k}^{1,N}, \ldots, \widehat{X}_{t_k}^{N,N}\right), \quad \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_{t_k}^{i,N}}(dx), \\ \mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}) = \min\left\{h(\widehat{X}_{t_k}^{1,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}), \ldots, h(\widehat{X}_{t_k}^{N,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\right\}, \quad t_{k+1} = t_k + \mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\Delta, \\ \widehat{X}_{t_{k+1}}^{i,N} = \widehat{X}_{t_k}^{i,N} + b(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(t_{k+1}-t_k) + \sigma_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(W_{t_{k+1}}^i - W_{t_k}^i) + c_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(Z_{t_{k+1}}^i - Z_{t_k}^i). \end{cases} \tag{21}$$

Remark 4.1. It follows from assumptions **A1**–**A5** that there exist positive constants L, β_1 and β_2 such that

T1. $\dfrac{1}{h(x,\mu)} \le L\left(1+|x|^{\beta_1}+\mathcal{W}_2^{\beta_2}(\mu,\delta_0)\right)$ and $|b(x,\mu)|\,(1+|b(x,\mu)|)\,h(x,\mu) \le L$;
T2. $\langle x, b(x,\mu)-b(0,\delta_0)\rangle \le L\left(|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right)$;
T3. $|\sigma_\Delta(x,\mu)|\,(1+|x|) \le \dfrac{1}{\sqrt{\Delta}}$; $|c_\Delta(x,\mu)|\,(1+|x|) \le \dfrac{1}{\sqrt{\Delta}}$; $|b(x,\mu)||c_\Delta(x,\mu)| \le \dfrac{1}{\sqrt{\Delta}}$;
T4. $|\sigma_\Delta(x,\mu)| \le |\sigma(x,\mu)|$ and $|c_\Delta(x,\mu)| \le |c(x,\mu)| \le L_3\,(1+|x|+\mathcal{W}_2(\mu,\delta_0))$;
T5. $|\sigma(x,\mu)-\sigma_\Delta(x,\mu)| \le \sqrt{\Delta}|\sigma(x,\mu)|^2(1+|x|)$; $|c(x,\mu)-c_\Delta(x,\mu)| \le \sqrt{\Delta}|c(x,\mu)|^2\,(1+|x|+|b(x,\mu)|)$,

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Next, we show that the tamed-adaptive Euler-Maruyama approximation scheme (21) is well-defined.

Proposition 4.2. *Let (t_k) be the sequence of stopping times defined in (21). Then, $\lim_{k\to+\infty} t_k = +\infty$ almost surely.*

Proof. For all $i \in \{1,\dots,N\}$ and $H>0$, we define the tamed-adaptive Euler-Maruyama discretization of equation (14) as follows

$$\begin{cases} t_0^H = 0, \quad \widehat{X}_0^{i,N,H} = x_0, \quad t_{k+1}^H = t_k^H + \mathbf{h}^H(\widehat{\boldsymbol{X}}_{t_k^H}^{N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})\Delta, \\ \widehat{X}_{t_{k+1}^H}^{i,N,H} = \widehat{X}_{t_k^H}^{i,N,H} + b_H(\widehat{X}_{t_k^H}^{i,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})(t_{k+1}^H - t_k^H) + \sigma_\Delta(\widehat{X}_{t_k^H}^{i,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})(W_{t_{k+1}^H}^i - W_{t_k^H}^i) \\ \qquad\qquad + c_\Delta(\widehat{X}_{t_k^H}^{i,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})(Z_{t_{k+1}^H}^i - Z_{t_k^H}^i), \end{cases} \tag{22}$$

where

$$\widehat{\boldsymbol{X}}_{t_k}^{N,H} = \left(\widehat{X}_{t_k}^{1,N,H}, \dots, \widehat{X}_{t_k}^{N,N,H}\right), \quad \mu_{t_k}^{\widehat{\boldsymbol{X}}^{N,H}}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_{t_k}^{i,N,H}}(dx),$$

$$\mathbf{h}^H(\widehat{\boldsymbol{X}}_{t_k^H}^N, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}}) = \min\left\{h^H(\widehat{X}_{t_k^H}^{1,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}}), \dots, h^H(\widehat{X}_{t_k^H}^{N,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})\right\},$$

$$h^H(x,\mu) = \begin{cases} h(x,\mu) & \text{if } |x|^{\beta_1} + \mathcal{W}_2^{\beta_2}(\mu,\delta_0) \le H, \\ \dfrac{1}{1+H} & \text{if } |x|^{\beta_1} + \mathcal{W}_2^{\beta_2}(\mu,\delta_0) > H, \end{cases}$$

$$b_H(x,\mu) = \begin{cases} b(x,\mu) & \text{if } |x|^{\beta_1} + \mathcal{W}_2^{\beta_2}(\mu,\delta_0) \le H, \\ \dfrac{x}{1+|x|^2} + b(0,\delta_0) & \text{if } |x|^{\beta_1} + \mathcal{W}_2^{\beta_2}(\mu,\delta_0) > H, \end{cases}$$

for $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Then, it can be checked that for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

(1) $|b_H(x,\mu)|h^H(x,\mu) \le C_0$ and $|b_H(x,\mu)|^2 h^H(x,\mu) \le C_0$,
(2) $\langle x, b_H(x,\mu) - b(0,\delta_0)\rangle \le C_0\left(|x|^2 + \mathcal{W}_2^2(\mu,\delta_0)\right)$ due to **T2**,

for some other positive constant C_0. Moreover, from **T1**, we have $h^H(x,\mu)\Delta \ge \dfrac{\min\{1,L^{-1}\}\Delta}{1+H}$, which implies that $t_{k+1}^H - t_k^H \ge \frac{\min\{1,L^{-1}\}\Delta}{1+H}$. Therefore, $\lim_{k\to+\infty} t_k^H = +\infty$ a.s.

Now, we define by $\underline{t}^H := \max\left\{t_k^H : t_k^H \le t\right\}$ the nearest time point before t. The continuous interpolant process is defined by

$$\begin{aligned}
\widehat{X}_t^{i,N,H} :&= \widehat{X}_{\underline{t}^H}^{i,N,H} + b_H\left(\widehat{X}_{\underline{t}^H}^{i,N,H}, \mu_{\underline{t}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)(t-\underline{t}^H) + \sigma_\Delta\left(\widehat{X}_{\underline{t}^H}^{i,N,H}, \mu_{\underline{t}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\left(W_t^i - W_{\underline{t}^H}^i\right) \\
&\quad + c_\Delta\left(\widehat{X}_{\underline{t}^H}^{i,N,H}, \mu_{\underline{t}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\left(Z_t^i - Z_{\underline{t}^H}^i\right) \\
&= x_0 + \int_0^t b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) ds + \int_0^t \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) dW_s^i \\
&\quad + \int_0^t \int_{\mathbb{R}_0^d} c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) z\widetilde{N}^i(ds,dz).
\end{aligned} \tag{23}$$

Using Itô's formula, we get

$$\begin{aligned}
|\widehat{X}_t^{i,N,H}|^2 &\le |x_0|^2 + \int_0^t \Bigg(2\left\langle \widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}, b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\right\rangle \\
&\quad + 2\left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) - b(0,\delta_0)\right\rangle \\
&\quad + 2\left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, b(0,\delta_0)\right\rangle + \left|\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\right|^2 + \left|c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\right|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz)\Bigg) ds + \mathcal{M}_t \\
&\le |x_0|^2 + \int_0^t \Bigg(2\left|\widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}\right| \left|b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\right| + 2C_0\left(|\widehat{X}_{\underline{s}^H}^{i,N,H}|^2 + \mathcal{W}_2^2(\mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}, \delta_0)\right) + |\widehat{X}_{\underline{s}^H}^{i,N,H}|^2 \\
&\quad + |b(0,\delta_0)|^2 + \frac{L^2}{\Delta} + \frac{L^2}{\Delta}\int_{\mathbb{R}_0^d} |z|^2\nu(dz)\Bigg) ds + \mathcal{M}_t,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\mathcal{M}_t =& 2\int_0^t \left\langle \widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}, \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) dW_s^i\right\rangle + 2\int_0^t \left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) dW_s^i\right\rangle \\
&+ \int_0^t \int_{\mathbb{R}_0^d} \left(\left|c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) z\right|^2 + 2\left\langle \widehat{X}_{s-}^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}, c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) z\right\rangle\right) \widetilde{N}^i(ds,dz) \\
&+ 2\int_0^t \int_{\mathbb{R}_0^d} \left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\mathbf{X}}^{N,H}}\right) z\right\rangle \widetilde{N}^i(ds,dz).
\end{aligned}$$

Now, we define $\tau_R := \inf\{t > 0 : \max_{i\in\{1,\dots,N\}} |\widehat{X}_t^{i,N,H}| > R\}$ for each $R > 0$ and $\tau := s \wedge \tau_R$. On the one hand, using equation (23), **T3**, the isometry property of stochastic integrals and the fact that $|b_H(x,\mu)|h^H(x,\mu) \le C_0$, we get

$$\begin{aligned}
\mathbb{E}\left[\left|\widehat{X}_\tau^{i,N,H} - \widehat{X}_{\underline{\tau}^H}^{i,N,H}\right|^2\right] \le 3\Bigg(&\mathbb{E}\left[\left|b_H\left(\widehat{X}_{\underline{\tau}^H}^{i,N,H}, \mu_{\underline{\tau}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\right|^2 (\tau - \underline{\tau}^H)^2\right] \\
&+ \left|\sigma_\Delta\left(\widehat{X}_{\underline{\tau}^H}^{i,N,H}, \mu_{\underline{\tau}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\right|^2 \left|W_\tau^i - W_{\underline{\tau}^H}^i\right|^2 + \left|c_\Delta\left(\widehat{X}_{\underline{\tau}^H}^{i,N,H}, \mu_{\underline{\tau}^H}^{\widehat{\mathbf{X}}^{N,H}}\right)\right|^2 \left|Z_\tau^i - Z_{\underline{\tau}^H}^i\right|^2\Bigg)
\end{aligned}$$

$$\leq 3\left(C_0^2\Delta^2 + \frac{L^2}{\Delta}\mathbb{E}\left[\tau - \underline{\tau}^H\right] + \frac{L^2}{\Delta}\int\limits_{\mathbb{R}_0^d}|z|^2\nu(dz)\mathbb{E}\left[\tau - \underline{\tau}^H\right]\right) \leq 3\left(C_0^2\Delta^2 + L^2 + L^2\int\limits_{\mathbb{R}_0^d}|z|^2\nu(dz)\right). \tag{25}$$

On the other hand, **T3** yields to

$$\begin{aligned}
&\left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}}, \quad \left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}}, \\
&\left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right|\left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}}, \quad \left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right|\left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}}.
\end{aligned} \tag{26}$$

Therefore, all the stochastic integrals to the Brownian motion and the compensated Poisson random measure above are square martingales. Thus, their moments are equal to zero.

In all that follows, the following classical moment estimates for the increment of Brownian motion W and Lévy process Z will be useful

$$\mathbb{E}\left[\left|W_t - W_{\underline{t}^H}\right|^r \left|\mathcal{F}_{\underline{t}^H}\right.\right] \leq C_r(t - \underline{t}^H)^{r/2}, \quad \mathbb{E}\left[\left|Z_t - Z_{\underline{t}^H}\right|^r \left|\mathcal{F}_{\underline{t}^H}\right.\right] \leq C_r(t - \underline{t}^H), \tag{27}$$

for any $t > 0$, $r \geq 1$ and some positive constant C_r.

Using **T3**, equation (23), moment properties of the Brownian motion, the isometry property of stochastic integrals, and the fact that $|b_H(x,\mu)|^2 h^H(x,\mu) \leq C_0$, we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_s - \widehat{X}^{i,N,H}_{\underline{s}^H}\right|\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \left|\mathcal{F}_{\underline{s}^H}\right.\right] \\
&\leq \mathbb{E}\Bigg[\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|^2(s - \underline{s}^H) + \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|\left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|\left|W^i_s - W^i_{\underline{s}^H}\right| \\
&\quad + \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|\left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|\left|Z^i_s - Z^i_{\underline{s}^H}\right| \left|\mathcal{F}_{\underline{s}^H}\right.\Bigg] \\
&\leq \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|^2(s - \underline{s}^H) + \frac{L}{\sqrt{\Delta}}\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|\sqrt{s - \underline{s}^H} \\
&\quad + \frac{L}{\sqrt{\Delta}}\left(\int\limits_{\mathbb{R}_0^d}|z|^2\nu(dz)\right)^{1/2}\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|\sqrt{s - \underline{s}^H} \\
&\leq C_0\Delta + L\sqrt{C_0} + L\sqrt{C_0}\left(\int\limits_{\mathbb{R}_0^d}|z|^2\nu(dz)\right)^{1/2}.
\end{aligned} \tag{28}$$

This, combined with $\mathbb{E}\left[\mathcal{W}_2^2(\mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}, \delta_0)\mathbf{1}_{s\leq\tau_R}\right] = \mathbb{E}\left[|\widehat{X}^{i,N,H}_{\underline{s}^H}|^2\mathbf{1}_{s\leq\tau_R}\right]$ for $i \in \{1,\ldots,N\}$, yields that for any $t \in (0,T]$,

$$\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{t\wedge\tau_R}\right|^2\right] \leq |x_0|^2 + \int\limits_0^t C\left(L, \Delta, b(0,\delta_0), \mu_2\right)\left(1 + \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right|^2\mathbf{1}_{s\leq\tau_R}\right]\right)ds,$$

where $\mu_2 := \int_{\mathbb{R}_0^d}|z|^2\nu(dz)$.

Next, using equation (23) and (25), we get

$$\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right|^2\mathbf{1}_{s\leq\tau_R}\right] \leq 2\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_s\right|^2\mathbf{1}_{s\leq\tau_R}\right] + 2\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_s - \widehat{X}^{i,N,H}_{\underline{s}^H}\right|^2\right]$$

$$\leq 2\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{s\wedge\tau_R}\right|^2\right] + 6\left(C_0^2\Delta^2 + L^2 + L^2\mu_2\right).$$

This implies that for any $t \in (0,T]$,

$$\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{t\wedge\tau_R}\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right)\left(1 + \int_0^t \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{s\wedge\tau_R}\right|^2\right] ds\right),$$

which, together with Gronwall's inequality, yields that for any $t \in (0,T]$,

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{t\wedge\tau_R}\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right).$$

Then, using Markov's inequality, we obtain that

$$\begin{aligned}\mathbb{P}(\tau_R < T) &\leq \sum_{i=1}^N \mathbb{P}(|\widehat{X}^{i,N,H}_{T\wedge\tau_R}| > R) = N\mathbb{P}(|\widehat{X}^{1,N,H}_{T\wedge\tau_R}| > R)\\ &\leq N\frac{\mathbb{E}[|\widehat{X}^{1,N,H}_{T\wedge\tau_R}|^2]}{R^2} \leq N\frac{C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right)}{R^2},\end{aligned}$$

which tends to zero as $R \uparrow \infty$. Therefore, $\tau_R \uparrow \infty$ as $R \uparrow \infty$. Then due to Fatou's lemma, we get

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{t}\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right). \tag{29}$$

Now, from (24), (28), (29) and the fact that $\mathbb{E}\left[\mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}, \delta_0)\right] = \mathbb{E}\left[|\widehat{X}^{i,N,H}_{\underline{s}^H}|^2\right]$, we get that

$$\max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]}\left|\widehat{X}^{i,N,H}_{t}\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right) =: \overline{C_0}. \tag{30}$$

Observe that

$$\begin{aligned}\{t_k \leq T\} \subset &\left\{t_k \leq T, \max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]}\left(\left|\widehat{X}^{i,N,H}_t\right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right)\right) \leq \frac{H}{2}\right\}\\ &\cup\left\{\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]}\left(\left|\widehat{X}^{i,N,H}_t\right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right)\right) > \frac{H}{2}\right\}\\ &\subset \left\{t_k^H \leq T\right\} \cup \left\{\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]}\left|\widehat{X}^{i,N,H}_t\right|^{\beta_1} > \frac{H}{4}\right\} \cup \left\{\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]}\mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right) > \frac{H}{4}\right\}.\end{aligned}$$

Then, using $\mathbb{E}\left[\sup_{t\in[0,T]}\mathcal{W}_2^2(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0)\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]}|\widehat{X}^{i,N,H}_t|^2\right]$ for $i \in \{1,\ldots,N\}$ and Markov's inequality, we get that for any $H > 0$,

$$\begin{aligned}\mathbb{P}(t_k \leq T) &\leq \mathbb{P}\left(t_k^H \leq T\right) + \left(\frac{4}{H}\right)^{2/\beta_1}\sum_{i=1}^N \mathbb{E}\left[\sup_{t\in[0,T]}\left|\widehat{X}^{i,N,H}_t\right|^2\right] + \left(\frac{4}{H}\right)^{2/\beta_2}\sum_{i=1}^N \mathbb{E}\left[\sup_{t\in[0,T]}\left|\widehat{X}^{i,N,H}_t\right|^2\right]\\ &\leq \mathbb{P}\left(t_k^H \leq T\right) + \left(\left(\frac{4}{H}\right)^{2/\beta_1} + \left(\frac{4}{H}\right)^{2/\beta_2}\right)N\overline{C_0}.\end{aligned}$$

Then, let $k \uparrow \infty$ and recall that $\lim_{k \to +\infty} t_k^H = +\infty$ a.s., we have that for any $H > 0$,

$$\limsup_{k \to \infty} \mathbb{P}(t_k \leq T) \leq \left(\left(\frac{4}{H}\right)^{2/\beta_1} + \left(\frac{4}{H}\right)^{2/\beta_2} \right) N\overline{C_0}.$$

Then, letting $H \uparrow \infty$, we get $\lim_{k\to\infty} \mathbb{P}(t_k \leq T) = 0$. Therefore, $t_k \to \infty$ in probability as $k \uparrow \infty$. Since $(t_k)_{k\geq 0}$ is an increasing sequence, we have $\lim_{k \to +\infty} t_k = +\infty$, a.s. Thus, the result follows. □

Remark 4.3. The fact that t_k goes to infinity is a very important and difficult step in building an adaptive scheme. Our proof of this is quite different from the proof in [10]. Indeed, in [10], the auxiliary process $\widehat{X}$ is constructed by projection, which makes its analysis very difficult in the case of McKean-Vlasov SDE. In our proof, $\widehat{X}$ is formulated as an Itô process, allowing us to apply the Itô formula to $|\widehat{X}|^2$. This fact greatly simplifies our proof.

Now we define by $\underline{t} := \max\{t_n : t_n \leq t\}$ the nearest time point before t, and by $N_t := \max\{n : t_n \leq t\}$ the number of timesteps approximation up to time t. Observe that $\underline{t}$ is a stopping time. Thus, we define the standard continuous interpolation as

$$\widehat{X}_t^{i,N} = \widehat{X}_{\underline{t}}^{i,N} + b\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)(t - \underline{t}) + \sigma_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)(W_t^i - W_{\underline{t}}^i) + c_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)(Z_t^i - Z_{\underline{t}}^i). \quad (31)$$

Hence, $\widehat{X}^{i,N} = (\widehat{X}_t^{i,N})_{t\geq 0}$ is the solution to the following SDE with jumps

$$d\widehat{X}_t^{i,N} = b\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)dt + \sigma_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)dW_t^i + c_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)dZ_t^i, \qquad \widehat{X}_0^{i,N} = x_0. \quad (32)$$

4.2. Moments of the tamed-adaptive Euler-Maruyama scheme

We now state the first estimate on the moments of $\widehat{X}^{i,N} = (\widehat{X}_t^{i,N})_{t\geq 0}$.

Lemma 4.4. *For any $p \in [1, 2p_0]$ and $T > 0$, there exists a positive constant $C\left(p, x_0, L, \Delta, b(0,\delta_0), \mu_2, \mu_{p/2}, T\right)$ with $\mu_{p/2} := \int_{\mathbb{R}_0^d} |z|^{p/2}\nu(dz)$ such that*

$$\max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|^p\right] \leq C\left(p, x_0, L, \Delta, b(0,\delta_0), \mu_2, \mu_{p/2}, T\right).$$

Proof. Recall that the process $\widehat{X}^{i,N,H} = (\widehat{X}_t^{i,N,H})_{t\geq 0}$ is defined in (22) and (23). Using Markov's inequality, the estimate $\mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{W}_2^2(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0)\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H}|^2\right]$ and (30), we obtain that for any $T > 0$, $i \in \{1, \ldots, N\}$ and $H > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} |\widehat{X}_t^{i,N}| \neq \sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H}|\right) \leq \mathbb{P}\left(\sup_{t\in[0,T]} \left(\left|\widehat{X}_t^{i,N,H}\right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right)\right) > H\right) \\
&\leq \mathbb{P}\left(\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^{\beta_1} > \frac{H}{2}\right) + \mathbb{P}\left(\sup_{t\in[0,T]} \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right) > \frac{H}{2}\right) \\
&\leq \left(\frac{2}{H}\right)^{2/\beta_1} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^2\right] + \left(\frac{2}{H}\right)^{2/\beta_2} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^2\right] \leq \left(\left(\frac{2}{H}\right)^{2/\beta_1} + \left(\frac{2}{H}\right)^{2/\beta_2}\right)\overline{C_0},
\end{aligned}$$

which tends to zero as $H \uparrow \infty$. This implies that $\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H}| \to \sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|$ in probability as $H \uparrow \infty$. Thus, for any $T > 0$ and $i \in \{1, \dots, N\}$, there exists a sequence $\{H_n\}_{n\geq 1}$ that tends to infinity such that $\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H_n}| \to \sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|$ a.s. as $n \uparrow \infty$.

Now, from (24), we have that for any $t > 0$, $i \in \{1, \dots, N\}$ and $H > 0$,

$$
\begin{aligned}
|\widehat{X}_t^{i,N,H}|^2 \leq |x_0|^2 + \int_0^T \Bigg(& 2\left|\widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}\right| \left|b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right| + 2C_0\mathcal{W}_2^2(\mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0) \\
& + |b(0,\delta_0)|^2 + \frac{L^2}{\Delta} + \frac{L^2}{\Delta}\int_{\mathbb{R}_0^d} |z|^2\nu(dz)\Bigg) ds + 2\int_0^t \left\langle \widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}, \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right) dW_s^i\right\rangle \\
& + 2\int_0^t \left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right) dW_s^i\right\rangle \\
& + \int_0^t \int_{\mathbb{R}_0^d} \left(\left| c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right) z\right|^2 + 2\left\langle \widehat{X}_{s-}^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}, c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right) z\right\rangle\right) \widetilde{N}^i(ds,dz) \\
& + 2\int_0^t \int_{\mathbb{R}_0^d} \left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right) z\right\rangle \widetilde{N}^i(ds,dz) + (2C_0+1)\int_0^t |\widehat{X}_{\underline{s}^H}^{i,N,H}|^2 ds.
\end{aligned} \tag{33}
$$

First, using the similar argument as in (28), we get

$$
\begin{aligned}
& \mathbb{E}\left[\left|\widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}\right|^{p/2} \left|b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right|^{p/2} \Big| \mathcal{F}_{\underline{s}^H}\right] \\
& \leq C(p)\left(C_0^{p/2}\Delta^{p/2} + L^{p/2}C_0^{p/4} + \left(\frac{L}{\sqrt{\Delta}}\right)^{p/2} \int_{\mathbb{R}_0^d} |z|^{p/2}\nu(dz)\right).
\end{aligned} \tag{34}
$$

Second, using the similar argument as in (25), we have

$$
\mathbb{E}\left[\left|\widehat{X}_\tau^{i,N,H} - \widehat{X}_{\underline{\tau}^H}^{i,N,H}\right|^{p/2}\right] \leq C_p\left(C_0^{p/2}\Delta^{p/2} + L^{p/2} + \left(\frac{L}{\sqrt{\Delta}}\right)^{p/2} \int_{\mathbb{R}_0^d} |z|^{p/2}\nu(dz)\right). \tag{35}
$$

Therefore, from (33), (34), (35), (26), the estimate $\mathbb{E}\left[\mathcal{W}_2^2(\mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0)\right] = \mathbb{E}\left[|\widehat{X}_{\underline{s}^H}^{i,N,H}|^2\right] \leq \overline{C_0}$ and the Burkholder-Davis-Gundy inequality ([2, Theorem 4.4.23] and [33, Proposition 2.2]), we get that for any $t \in (0,T]$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{u\in[0,t]} |\widehat{X}_u^{i,N,H}|^p\right] & \leq C\left(p, x_0, L, \Delta, b(0,\delta_0), \mu_2, \mu_{p/2}, T\right) + (2C_0+1)^{p/2}\mathbb{E}\left[\left(\int_0^t |\widehat{X}_{\underline{s}^H}^{i,N,H}|^2 ds\right)^{p/2}\right] \\
& \leq \widetilde{C}_0 + (2C_0+1)^{p/2} t^{p/2-1} \int_0^t \mathbb{E}\left[\sup_{u\in[0,s]} |\widehat{X}_u^{i,N,H}|^p\right] ds,
\end{aligned}
$$

where $\widetilde{C}_0 := C\left(p, x_0, L, \Delta, b(0,\delta_0), \mu_2, \mu_{p/2}, T\right)$ with $\mu_{p/2} := \int_{\mathbb{R}_0^d} |z|^{p/2}\nu(dz)$.

This, combined with Gronwall's inequality, deduces that

$$\max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H}|^p\right] \leq \widetilde{C}_1, \tag{36}$$

where the constant $\widetilde{C}_1$ does not depend on H.

Therefore, choosing $H = H_n$ in (36) and letting $n \uparrow \infty$, combined with Fatou's lemma and the fact that $\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H_n}| \to \sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|$ a.s. as $n \uparrow \infty$, we obtain that $\max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|^p\right] \leq \widetilde{C}_1$, which finishes the desired proof. □

In the following, we state an estimate for L^2-norm of the approximate solution.

Lemma 4.5. *There exists a positive constant $C = C(x_0, \gamma_1, \gamma_2, \eta, L, L_3)$ which depends neither on Δ nor on t such that for any $t \geq 0$,*

$$\max_{i\in\{1,\ldots,N\}} \left(\mathbb{E}\left[|\widehat{X}_t^{i,N}|^2\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}^{i,N}|^2\right]\right) \leq \begin{cases} Ce^{2\gamma t} & \text{if } \gamma > 0, \\ C(1+t) & \text{if } \gamma = 0, \\ C & \text{if } \gamma < 0, \end{cases}$$

where $\gamma = \gamma_1 + \gamma_2$.

Proof. Proceeding as in (12) by applying Itô's formula to $e^{-2\gamma t}|\widehat{X}_t^{i,N}|^2$, we get that for any $i \in \{1,\ldots,N\}$,

$$\begin{aligned} e^{-2\gamma t}|\widehat{X}_t^{i,N}|^2 \leq |x_0|^2 &+ 2\int_0^t e^{-2\gamma s}\Bigg(-\gamma|\widehat{X}_s^{i,N}|^2 + \langle \widehat{X}_s^{i,N}, b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\rangle + \frac{1}{2}\left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2 \\ &+ \frac{1}{2}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz)\Bigg)ds + 2\int_0^t e^{-2\gamma s}\left\langle \widehat{X}_s^{i,N}, \sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})dW_s^i\right\rangle \\ &+ \int_0^t\int_{\mathbb{R}_0^d} e^{-2\gamma s}\left(2\left\langle \widehat{X}_{\underline{s}-}^{i,N}, c_\Delta\left(\widehat{X}_{\underline{s}-}^{i,N}, \mu_{\underline{s}-}^{\widehat{\mathbf{X}}^N}\right)z\right\rangle + \left|c_\Delta\left(\widehat{X}_{\underline{s}-}^{i,N}, \mu_{\underline{s}-}^{\widehat{\mathbf{X}}^N}\right)z\right|^2\right)\widetilde{N}^i(ds,dz). \end{aligned} \tag{37}$$

First, using (31), we have

$$\begin{aligned} -\gamma|\widehat{X}_s^{i,N}|^2 \leq -\gamma\left|\widehat{X}_{\underline{s}}^{i,N}\right|^2 &+ 2|\gamma|\left|\widehat{X}_{\underline{s}}^{i,N}\right|\left|b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right|(s-\underline{s}) \\ &- 2\gamma\left\langle \widehat{X}_{\underline{s}}^{i,N}, \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)(W_s^i - W_{\underline{s}}^i) + c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)(Z_s^i - Z_{\underline{s}}^i)\right\rangle \\ &+ 3|\gamma|\Bigg(\left|b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2(s-\underline{s})^2 + \left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2 \left|W_s^i - W_{\underline{s}}^i\right|^2 \\ &+ \left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2 |Z_s^i - Z_{\underline{s}}^i|^2\Bigg). \end{aligned}$$

Then, using **T4**, **A4**, (27), and (20), we get

$$\max\left\{\left|\widehat{X}_{\underline{s}}^{i,N}\right|\left|b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|(s-\underline{s});\ \left|b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2(s-\underline{s})^2;\ \mathbb{E}\left[\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right|^2\left|W_s^i-W_{\underline{s}}^i\right|^2|\mathcal{F}_{\underline{s}}\right];\right.$$
$$\left.\mathbb{E}\left[\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right|^2\left|Z_s^i-Z_{\underline{s}}^i\right|^2|\mathcal{F}_{\underline{s}}\right]\right\}\le C\Delta,$$

which yields that

$$\mathbb{E}\left[-\gamma|\widehat{X}_s^{i,N}|^2\right]=\mathbb{E}\left[\mathbb{E}\left[-\gamma|\widehat{X}_s^{i,N}|^2|\mathcal{F}_{\underline{s}}\right]\right]\le\mathbb{E}\left[-\gamma|\widehat{X}_{\underline{s}}^{i,N}|^2\right]+C|\gamma|\Delta. \tag{38}$$

Moreover, using again (31), we have

$$\begin{aligned}\left\langle\widehat{X}_s^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle&=\left\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle+\left\langle\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle\\&\le\left\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle+\left|b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2(s-\underline{s})\\&\quad+\left\langle\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)(W_s^i-W_{\underline{s}}^i)+c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)(Z_s^i-Z_{\underline{s}}^i),b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle,\end{aligned}$$

which, combined with the fact that $|b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})|^2(s-\underline{s})\le\Delta$ due to (20), deduces that

$$\mathbb{E}\left[\left\langle\widehat{X}_s^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle\right]\le\mathbb{E}\left[\left\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle\right]+\Delta. \tag{39}$$

Thanks to Lemma 4.4, the stochastic integrals in (37) have zero expectations. Thus, using (37), (38), (39), **T4** and **A6**, and recall that $\gamma=\gamma_1+\gamma_2$, we obtain that

$$\begin{aligned}\mathbb{E}\left[e^{-2\gamma t}|\widehat{X}_t^{i,N}|^2\right]&\le|x_0|^2+2\int_0^t e^{-2\gamma s}\Bigg(\mathbb{E}\Bigg[-\gamma|\widehat{X}_{\underline{s}}^{i,N}|^2+\left\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle+\frac{1}{2}\left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2\\&\qquad+\frac{1}{2}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\Bigg]+C\Bigg)ds\\&\le|x_0|^2+2\int_0^t e^{-2\gamma s}\Big(-\gamma_2\mathbb{E}\left[|\widehat{X}_{\underline{s}}^{i,N}|^2\right]+\gamma_2\mathbb{E}\left[\mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N},\delta_0)\right]+\eta+C\Big)ds\\&=|x_0|^2+2(\eta+C)\int_0^t e^{-2\gamma s}ds,\end{aligned}$$

for some positive constant C, where we have used the equality $\mathbb{E}\left[\mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N},\delta_0)\right]=\mathbb{E}\left[|\widehat{X}_{\underline{s}}^{i,N}|^2\right]$. This yields to

$$\mathbb{E}\left[|\widehat{X}_t^{i,N}|^2\right]\le\begin{cases}\left(|x_0|^2+\dfrac{\eta+C}{\gamma}\right)e^{2\gamma t}-\dfrac{\eta+C}{\gamma} & \text{if}\quad\gamma\neq0,\\|x_0|^2+2(\eta+C)t & \text{if}\quad\gamma=0.\end{cases} \tag{40}$$

Next, from (31), we have

$$\widehat{X}_{\underline{t}}^{i,N}=\widehat{X}_t^{i,N}-b\left(\widehat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\widehat{\mathbf{X}}^N}\right)(t-\underline{t})-\sigma_\Delta\left(\widehat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\widehat{\mathbf{X}}^N}\right)(W_t^i-W_{\underline{t}}^i)-c_\Delta\left(\widehat{X}_{\underline{t}}^{i,N},\mu_{\underline{t}}^{\widehat{\mathbf{X}}^N}\right)(Z_t^i-Z_{\underline{t}}^i).$$

This, combined with **T4**, **A4**, (27) and (20), we get that for any $p > 1$,

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{t}}\right|^p\right] \leq 4^{p-1}\Bigg(&\mathbb{E}\left[\left|\widehat{X}^{i,N}_{t}\right|^p\right] + \mathbb{E}\left[\left|b(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}})(t-\underline{t})\right|^p\right] + \mathbb{E}\left[\left|\sigma_\Delta(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}})(W^i_t - W^i_{\underline{t}})\right|^p\right]\\
&+ \mathbb{E}\left[\left|c_\Delta(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}})(Z^i_t - Z^i_{\underline{t}})\right|^p\right]\Bigg)\\
\leq 4^{p-1}\Bigg(&\mathbb{E}\left[\left|\widehat{X}^{i,N}_{t}\right|^p\right] + C\Delta^p + C\Delta^{p/2} + C\Delta^{1\wedge p/2}\Bigg). \end{aligned} \tag{41}$$

Consequently, from (40) and (41) with $p = 2$, the result follows. □

To estimate L^p-norm of the approximate solution for $p > 2$, we need a series of preliminary lemmas.

Lemma 4.6. *Let p be a positive even integer. For any $a, b, c \in \mathbb{R}^d$, it holds that*

$$\begin{aligned}S =&|a+c|^p - |b+c|^p - |a|^p + |b|^p\\
\leq&\sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k}2^{j-k}|c|^{j+k}\Big[2^{p-2j-1}(|a-b|^{p-2j} + |b|^{p-2j})\sum_{\ell=1}^{j-k}\binom{j-k}{\ell}|b|^{j-k-\ell}|a-b|^\ell\\
&+\sum_{\ell=1}^{p-2j}\binom{p-2j}{\ell}|b|^{p-j-k-\ell}|a-b|^\ell\Big].\end{aligned}$$

Proof. We first note that $S = \left(|a|^2 + 2\langle a, c\rangle + |c|^2\right)^{p/2} - |a|^p - \left(|b|^2 + 2\langle b, c\rangle + |c|^2\right)^{p/2} + |b|^p$. Using the binomial theorem, we have

$$S = \sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k}2^{j-k}|c|^{2k}\Big(|a|^{p-2j}\langle a, c\rangle^{j-k} - |b|^{p-2j}\langle b, c\rangle^{j-k}\Big).$$

Next, we write

$$\begin{aligned}|a|^{p-2j}\langle a, c\rangle^{j-k} - |b|^{p-2j}\langle b, c\rangle^{j-k} = &|a|^{p-2j}\Big((\langle b, c\rangle + \langle a-b, c\rangle)^{j-k} - \langle b, c\rangle^{j-k}\Big)\\
&+\Big((|b| + (|a|-|b|))^{p-2j} - |b|^{p-2j}\Big)\langle b, c\rangle^{j-k}.\end{aligned}$$

Using the binomial theorem, the estimates $|a|^{p-2j} \leq 2^{p-2j-1}(|a-b|^{p-2j} + |b|^{p-2j})$, $|\langle a-b, c\rangle| \leq |a-b||c|$, and $|\langle b, c\rangle| \leq |b||c|$, we obtain the desired result. □

Lemma 4.7. *For any even integer $p \in (0, p_0]$, there exists a positive constant C_p such that for any $s > 0$, $i \in \{1, \ldots, N\}$ and $\lambda \in \mathbb{R}$,*

a) $\mathbb{E}\left[-\lambda|\widehat{X}^{i,N}_s|^p\middle|\mathcal{F}_{\underline{s}}\right] \leq -\lambda|\widehat{X}^{i,N}_{\underline{s}}|^p + C_p|\lambda|\sum_{j=0}^{p-2}|\widehat{X}^{i,N}_{\underline{s}}|^j.$

b) $\mathbb{E}\left[|\widehat{X}^{i,N}_s|^{p-2}\left\langle\widehat{X}^{i,N}_s, b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle\middle|\mathcal{F}_{\underline{s}}\right] \leq |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\left\langle\widehat{X}^{i,N}_{\underline{s}}, b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle + C_p\sum_{j=0}^{p-2}|\widehat{X}^{i,N}_{\underline{s}}|^j.$

c) $\mathbb{E}\left[|\widehat{X}^{i,N}_s|^{p-4}|(\widehat{X}^{i,N}_s)^\mathsf{T}\sigma_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})|^2\middle|\mathcal{F}_{\underline{s}}\right] \leq |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}|\sigma_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})|^2 + C_p\sum_{j=0}^{p-3}|\widehat{X}^{i,N}_{\underline{s}}|^j.$

Proof. First, by using **T4**, (20), Burkholder-Davis-Gundy's inequality ([2, Theorem 4.4.23] and [33, Proposition 2.2]), (27) and **A4**, we get that for all $2 \leq j \leq p$,

$$
\begin{aligned}
&\max\left\{|\widehat{X}_{\underline{s}}^{i,N}|\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right||s-\underline{s}|;\ \mathbb{E}\left[|\widehat{X}_{\underline{s}}^{i,N}|\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right||W_s^i-W_{\underline{s}}^i|\,|\mathcal{F}_{\underline{s}}\right];\right.\\
&\left.\quad\mathbb{E}\left[|\widehat{X}_{\underline{s}}^{i,N}|\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right||Z_s^i-Z_{\underline{s}}^i|\,|\mathcal{F}_{\underline{s}}\right]\right\}\leq C\sqrt{\Delta},\\
&\max\left\{\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right|^j|s-\underline{s}|^j;\ \mathbb{E}\left[\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right|^j|W_s^i-W_{\underline{s}}^i|^j\,|\mathcal{F}_{\underline{s}}\right];\right.\\
&\left.\quad\mathbb{E}\left[\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)\right|^j|Z_s^i-Z_{\underline{s}}^i|^j\,|\mathcal{F}_{\underline{s}}\right]\right\}\leq C\Delta.
\end{aligned}
\tag{42}
$$

These estimates of Lemma 4.7 follow from the binomial theorem, (31) and (42). □

Lemma 4.8. *For any even integer $p \in (0, p_0]$, $s > 0$, and $z \in \mathbb{R}^d$, it holds that*

$$
\begin{aligned}
&\mathbb{E}\left[\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p-\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p\right)-\left(|\widehat{X}_s^{i,N}|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p\right)|\mathcal{F}_{\underline{s}}\right]\\
&\leq \widehat{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|,|z|\right)+\sum_{j=1}^{p/2}\sum_{k=0}^{j}C_{j,k}|z|^{j+k}\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N},\delta_0)\\
&\quad\times\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-2}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-2}+\sum_{\ell=2}^{j-k}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\right)+\sum_{\ell=2}^{p-2j}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\right),
\end{aligned}
$$

where $(C_{j,k})$ are some positive constants, $\widehat{Q}_{p-2}(|\widehat{X}_{\underline{s}}^{i,N}|,|z|)$ is a polynomial in $|\widehat{X}_{\underline{s}}^{i,N}|$ of degree $p-2$.

Proof. By using Lemma 4.6, we have

$$
\begin{aligned}
&\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p-\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p\right)-\left(|\widehat{X}_s^{i,N}|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p\right)\\
&\leq\sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k}2^{j-k}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^{j+k}\left(2^{p-2j-1}\sum_{\ell=1}^{j-k}\binom{j-k}{\ell}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{p-2j+\ell}\right.\\
&\quad\left.+2^{p-2j-1}\sum_{\ell=1}^{j-k}\binom{j-k}{\ell}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}+\sum_{\ell=1}^{p-2j}\binom{p-2j}{\ell}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\right).
\end{aligned}
$$

Then, using **T4**, (31) and the estimate (42) we obtain the desired result. □

Lemma 4.9. *Let υ be a positive constant satisfying $N \geq \left(\frac{\max\{3L_3,1\}}{2\upsilon}\right)^2$. Then, for any even integer $p \in (0, p_0]$, $s > 0$, and $z \in \mathbb{R}^d$, it holds*

$$
\begin{aligned}
&\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p-p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right\rangle\\
&\leq\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}p\left(\frac{|z|^2}{2}+L_3^{-2}\left((1+|z|(L_3+\upsilon))^{p-1}-|z|(L_3+\upsilon)-1\right)\left(|z|\left(\frac{L_3}{2}+\upsilon\right)+\upsilon\right)\right)\\
&\quad+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right),
\end{aligned}
$$

where $\widehat{U}^{i,N}_{\underline{s}} := \frac{1}{\sqrt{N}} \sum_{j=1;j\neq i}^{N} \left|\widehat{X}^{j,N}_{\underline{s}}\right|$, *and*

$$Q_p\left(p-2, 2k-\ell, |\widehat{X}^{i,N}_{\underline{s}}|, 1+\widehat{U}^{i,N}_{\underline{s}}\right) := 2L_3^2\left(\left(1+\frac{1}{\sqrt{N}}\right)^2|\widehat{X}^{i,N}_{\underline{s}}|^2 + \left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^2\right)\binom{k}{\ell}2^\ell L_3^{2k-\ell-2}$$
$$\times\left(\left(k-\frac{\ell}{2}-1\right)(1+\frac{1}{\sqrt{N}})^{2k-\ell-3}\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-4}\right.$$
$$\left.+\sum_{m=2}^{2k-\ell-2}\binom{2k-\ell-2}{m}(1+\frac{1}{\sqrt{N}})^{2k-\ell-2-m}\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^m|\widehat{X}^{i,N}_{\underline{s}}|^{p-2-m}\right).$$

Proof. Proceeding in the same way as in (7), we get

$$\left|\widehat{X}^{i,N}_{\underline{s}} + c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\right|^p - |\widehat{X}^{i,N}_{\underline{s}}|^p - p|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\left\langle\widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\right\rangle$$
$$\le \frac{p}{2}\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{p-2}\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\right|^2 + \sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}\binom{k}{\ell}2^\ell|\widehat{X}^{i,N}_{\underline{s}}|^{p-2k+\ell}\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^{2k-\ell}|z|^{2k-\ell}.$$

It follows from **T4** and the estimate $\mathcal{W}_2(\mu^{\widehat{\mathbf{X}}^N}_{\underline{s}}, \delta_0) \le \frac{1}{\sqrt{N}}\sum_{j=1}^{N}|\widehat{X}^{j,N}_{\underline{s}}|$ that

$$\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^{2k-\ell-2} \le L_3^{2k-\ell-2}\left(1+|\widehat{X}^{i,N}_{\underline{s}}| + \mathcal{W}_2(\mu^{\widehat{\mathbf{X}}^N}_{\underline{s}}, \delta_0)\right)^{2k-\ell-2}$$
$$\le L_3^{2k-\ell-2}\left((1+\frac{1}{\sqrt{N}})|\widehat{X}^{i,N}_{\underline{s}}| + 1 + \frac{1}{\sqrt{N}}\sum_{j=1;j\neq i}^{N}\left|\widehat{X}^{j,N}_{\underline{s}}\right|\right)^{2k-\ell-2}$$
$$= L_3^{2k-\ell-2}\left((1+\frac{1}{\sqrt{N}})^{2k-\ell-2}|\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-2} + (2k-\ell-2)\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)(1+\frac{1}{\sqrt{N}})^{2k-\ell-3}|\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-3}\right.$$
$$\left.+\sum_{m=2}^{2k-\ell-2}\binom{2k-\ell-2}{m}\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^m(1+\frac{1}{\sqrt{N}})^{2k-\ell-2-m}|\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-2-m}\right). \tag{43}$$

Using the estimate $\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)|\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-3} \le \frac{1}{2}\left(\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^2|\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-4} + |\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-2}\right)$, we get

$$\sum_{\ell=0}^{k}\binom{k}{\ell}2^\ell|\widehat{X}^{i,N}_{\underline{s}}|^{p-2k+\ell}\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^{2k-\ell}|z|^{2k-\ell}$$
$$\le\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\sum_{\ell=0}^{k}\binom{k}{\ell}2^\ell|z|^{2k-\ell}L_3^{2k-\ell-2}\left(1+\frac{1}{\sqrt{N}}\right)^{2k-\ell-3}\left(\frac{1}{\sqrt{N}}+k-\frac{\ell}{2}\right)$$
$$+\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^2\sum_{\ell=0}^{k}\binom{k}{\ell}2^\ell|z|^{2k-\ell}L_3^{2k-\ell-2}\left(\left(k-\frac{\ell}{2}-1\right)(1+\frac{1}{\sqrt{N}})^{2k-\ell-3}\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-4}\right.$$
$$\left.+\sum_{m=2}^{2k-\ell-2}\binom{2k-\ell-2}{m}(1+\frac{1}{\sqrt{N}})^{2k-\ell-2-m}\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^m|\widehat{X}^{i,N}_{\underline{s}}|^{p-2-m}\right).$$

Set $a = |z|L_3\left(1+\frac{1}{\sqrt{N}}\right)$. Note that

$$\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|z|^{2k-\ell}L_3^{2k-\ell-2}\left(1+\frac{1}{\sqrt{N}}\right)^{2k-\ell-3} = L_3^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\left(a^2+2a\right)^{k},$$

$$-\frac{1}{2}\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|z|^{2k-\ell}L_3^{2k-\ell-2}\left(1+\frac{1}{\sqrt{N}}\right)^{2k-\ell-3}\ell = -L_3^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}ka\left(a^2+2a\right)^{k-1}.$$

These facts imply that

$$\begin{aligned}&\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|z|^{2k-\ell}L_3^{2k-\ell-2}\left(1+\frac{1}{\sqrt{N}}\right)^{2k-\ell-3}\left(\frac{1}{\sqrt{N}}+k-\frac{\ell}{2}\right)\\ &= L_3^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\left(a^2+2a\right)^{k-1}\Big(\frac{a^2+2a}{\sqrt{N}}+k(a^2+a)\Big).\end{aligned}$$

Moreover, similar to the estimate (43), we get

$$\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2 \le 2L_3^2\left(\left(1+\frac{1}{\sqrt{N}}\right)^2|\widehat{X}_{\underline{s}}^{i,N}|^2+\left(1+\widehat{U}_{\underline{s}}^{i,N}\right)^2\right).$$

Therefore, we have

$$\begin{aligned}&\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|\widehat{X}_{\underline{s}}^{i,N}|^{p-2k+\ell}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^{2k-\ell}|z|^{2k-\ell}\\ &\le \left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}L_3^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\left(a^2+2a\right)^{k-1}\left(\frac{a^2+2a}{\sqrt{N}}+k(a^2+a)\right)\\ &\quad+\sum_{\ell=0}^{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right).\end{aligned}$$

Next, using $\sum_{k=2}^{p/2}\binom{p/2}{k}kz^{k-1} = \frac{p}{2}((1+z)^{p/2-1}-1)$ and $\sum_{k=2}^{p/2}\binom{p/2}{k}z^{k-1} = z^{-1}((1+z)^{p/2}-1-\frac{p}{2}z)$ with $z>0$, we get that

$$\begin{aligned}&\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}\binom{k}{\ell}2^{\ell}|\widehat{X}_{\underline{s}}^{i,N}|^{p-2k+\ell}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^{2k-\ell}|z|^{2k-\ell}\\ &\le \left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}L_3^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\\ &\quad\times\left(\frac{1}{\sqrt{N}}\left((1+a)^p-1-\frac{p}{2}(a^2+2a)\right)+(a^2+a)\frac{p}{2}\left((1+a)^{p-2}-1\right)\right)\\ &\quad+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right).\end{aligned}$$

Applying the inequality $(1+a)^p-1-\frac{p}{2}(a^2+2a)\le(a+1)^2\left((1+a)^{p-2}-1\right)$, with $p\ge 2$ and $a>0$, we get that for $\upsilon \ge \frac{1}{2\sqrt{N}}\max\{3L_3,1\}$,

$$
\begin{aligned}
&\left|\widehat{X}^{i,N}_{\underline{s}} + c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\right|^p - |\widehat{X}^{i,N}_{\underline{s}}|^p - p|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\left\langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\right\rangle \\
&\le \left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^2 |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\Bigg(\frac{p}{2}|z|^2 + L_3^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3} \\
&\quad \times \left(\frac{1}{\sqrt{N}}\Big((1+a)^p - 1 - \frac{p}{2}(a^2+2a)\Big) + (a^2+a)\frac{p}{2}\Big((1+a)^{p-2}-1\Big)\right)\Bigg) \\
&\quad + \sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2, 2k-\ell, |\widehat{X}^{i,N}_{\underline{s}}|, 1+\widehat{U}^{i,N}_{\underline{s}}\right) \\
&\le \left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^2 |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} p\left(\frac{|z|^2}{2} + L_3^{-2}\Big((1+a)^{p-1} - a - 1\Big)\Big(\frac{a+1}{p\sqrt{N}} + \frac{a}{2}\Big)\right) \\
&\quad + \sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2, 2k-\ell, |\widehat{X}^{i,N}_{\underline{s}}|, 1+\widehat{U}^{i,N}_{\underline{s}}\right) \\
&\le \left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})\right|^2 |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} p\left(\frac{|z|^2}{2} + L_3^{-2}\left((1+|z|(L_3+\upsilon))^{p-1} - |z|(L_3+\upsilon) - 1\right)\left(|z|\Big(\frac{L_3}{2}+\upsilon\Big)+\upsilon\right)\right) \\
&\quad + \sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2, 2k-\ell, |\widehat{X}^{i,N}_{\underline{s}}|, 1+\widehat{U}^{i,N}_{\underline{s}}\right),
\end{aligned}
$$

where we have used the fact that $a \le |z|(L_3+\upsilon)$, $\quad \dfrac{a+1}{p\sqrt{N}} + \dfrac{a}{2} \le |z|\Big(\frac{L_3}{2}+\upsilon\Big)+\upsilon$. $\square$

Lemma 4.10. *For any even integer $p \in (0, p_0]$, there exists a positive constant C_p such that for any $s > 0$ and $z \in \mathbb{R}^d$,*

$$
\begin{aligned}
&\mathbb{E}\left[|\widehat{X}^{i,N}_s|^{p-2}\langle \widehat{X}^{i,N}_s, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle - |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle \big| \mathcal{F}_{\underline{s}}\right] \\
&\le C_p|z|\left(\sum_{j=0}^{p-2}|\widehat{X}^{i,N}_{\underline{s}}|^j + \mathcal{W}_2(\mu^{\widehat{\mathbf{X}}^N}_{\underline{s}}, \delta_0)\sum_{j=0}^{p-3}|\widehat{X}^{i,N}_{\underline{s}}|^j\right).
\end{aligned}
$$

Proof. By using the binomial theorem and (31), we have

$$
\begin{aligned}
&|\widehat{X}^{i,N}_s|^{p-2}\langle \widehat{X}^{i,N}_s, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle - |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle \\
&= \langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle\left(|\widehat{X}^{i,N}_s|^{p-2} - |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\right) + \langle \widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle|\widehat{X}^{i,N}_s|^{p-2} \\
&= \langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle\sum_{j=1}^{p-2}\binom{p-2}{j}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2-j}\left(|\widehat{X}^{i,N}_s| - |\widehat{X}^{i,N}_{\underline{s}}|\right)^j \\
&\quad + \langle \widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle|\widehat{X}^{i,N}_{\underline{s}}|^{p-2} \\
&\quad + \langle \widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\mathbf{X}}^N}_{\underline{s}})z\rangle\sum_{j=1}^{p-2}\binom{p-2}{j}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2-j}\left(|\widehat{X}^{i,N}_s| - |\widehat{X}^{i,N}_{\underline{s}}|\right)^j.
\end{aligned}
$$

By applying (42) and **T4**, we obtain the desired result. $\square$

Proposition 4.11. *There exist constants $\widetilde{\gamma}_1 \in \mathbb{R}$, $\widetilde{\gamma}_2 > 0$, $\widetilde{\eta} \geq 0$, and $\upsilon > 0$, such that for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,*

$$\begin{aligned}&\langle x, b(x,\mu)\rangle + \frac{p_0-1}{2}|\sigma_\Delta(x,\mu)|^2 \\ &+ |c_\Delta(x,\mu)|^2 \int\limits_{\mathbb{R}_0^d} \left(\frac{|z|^2}{2} + \frac{1}{L_3^2}\Big((1+|z|(L_3+\upsilon))^{p_0-1} - 1 - |z|(L_3+\upsilon)\Big)\left(|z|\Big(\frac{L_3}{2}+\upsilon\Big)+\upsilon\right)\right)\nu(dz) \qquad (44)\\ &\leq \widetilde{\gamma}_1|x|^2 + \widetilde{\gamma}_2\mathcal{W}_2^2(\mu,\delta_0) + \widetilde{\eta}.\end{aligned}$$

Moreover, if $\gamma_1+\gamma_2<0$, we can choose $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}$ and υ such that $\widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$.

Proof. Based on **A6** and **T4**, it follows that (44) holds for $\upsilon = 0$ and $\widetilde{\gamma}_1 = \gamma_1$, $\widetilde{\gamma}_2 = \gamma_2$, $\widetilde{\eta} = \eta$. Note that in (44), the integral with respect to $\nu(dz)$ is finite for any $\upsilon > 0$. This fact, combined with **T4**, implies that for any $\upsilon > 0$, there exist constants $\widetilde{\gamma}_1 \in \mathbb{R}$, $\widetilde{\gamma}_2 > 0$, $\widetilde{\eta} \geq 0$ such that (44) holds. Next, we note that the left-hand side of (44) is continuous and increasing with respect to υ on the interval $[0,+\infty)$. Therefore, if **A6** holds with $\gamma_1+\gamma_2<0$, we can select υ to be sufficiently small such that (44) holds for some constants $\widetilde{\eta} \geq 0$, $\widetilde{\gamma}_1 \in \mathbb{R}$, and $\widetilde{\gamma}_2 > 0$, satisfying $\widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$. □

Remark 4.12.

i) If (44) holds for some $p_0 \geq 2$ then it also holds for any $p \in [2; p_0]$.
ii) It follows from **A6** and **T4** that if $p_0 = 2$ then (44) holds with $\widetilde{\gamma}_1 = \gamma_1$, $\widetilde{\gamma}_2 = \gamma_2$, $\widetilde{\eta} = \eta$.

Next, we are going to show how the moments of $\widehat{X}_t^{i,N}$ are bounded in terms of t.

Proposition 4.13. *Let $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\eta}$, and υ be defined as in Proposition 4.11. Assume that $N \geq \left(\frac{\max\{3L_3,1\}}{2\upsilon}\right)^2$. Then, for any positive $k \leq p_0/2$, there exists a positive constant $C = C(x_0, k, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, L, L_3, p_0)$ which depends neither on Δ nor on t such that for any $t \geq 0$,*

$$\max_{i\in\{1,\dots,N\}}\left(\mathbb{E}\left[|\widehat{X}_t^{i,N}|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}^{i,N}|^{2k}\right]\right) \leq \begin{cases} Ce^{2k\widetilde{\gamma}t} & \text{if } \widetilde{\gamma} > 0,\\ C(1+t)^k & \text{if } \widetilde{\gamma} = 0,\\ C & \text{if } \widetilde{\gamma} < 0,\end{cases} \tag{45}$$

where $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2$.

Proof. Using Hölder's inequality, it suffices to show (45) for a positive integer $k \leq p_0/2$. We are going to use the induction method. First, note that (45) is valid for $k = 1$ due to Remark 4.12 and Lemma 4.5.

Next, assume that (45) holds for any $k \leq k_0 \leq [p_0/2] - 1$. We wish to show that (45) still holds for $k = k_0 + 1$. For this, using Itô's formula for $e^{-p\lambda t}|\widehat{X}_t^{i,N}|^p$ with even integer $p := 2(k_0+1) \leq p_0$, we have

$$e^{-p\lambda t}\left|\widehat{X}_t^{i,N}\right|^p = |x_0|^p + \int\limits_0^t e^{-p\lambda s}\mathcal{R}_s ds + \mathcal{M}_t, \tag{46}$$

where

$$\begin{aligned}\mathcal{R}_s = &-p\lambda|\widehat{X}_s^{i,N}|^p + p|\widehat{X}_s^{i,N}|^{p-2}\left\langle \widehat{X}_s^{i,N}, b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle\\ &+ \frac{p}{2}|\widehat{X}_s^{i,N}|^{p-2}\left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2 + \frac{p(p-2)}{2}|\widehat{X}_s^{i,N}|^{p-4}\left|(\widehat{X}_s^{i,N})^\mathsf{T}\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2\end{aligned}$$

$$+\int_{\mathbb{R}_0^d}\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p-|\widehat{X}_s^{i,N}|^p-p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right\rangle\right)\nu(dz),$$

$$\mathcal{M}_t=p\int_0^t e^{-p\lambda s}|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})dW_s^i\right\rangle$$
$$+\int_0^t\int_{\mathbb{R}_0^d}e^{-p\lambda s}\left(\left|\widehat{X}_{s-}^{i,N}+c_\Delta(\widehat{X}_{\underline{s-}}^{i,N},\mu_{\underline{s-}}^{\widehat{\mathbf{X}}^N})z\right|^p-|\widehat{X}_{s-}^{i,N}|^p\right)\widetilde{N}^i(ds,dz).$$

It follows from Lemma 4.7 that

$$\begin{aligned}&\mathbb{E}\Big[-p\lambda|\widehat{X}_s^{i,N}|^p+p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle+\frac{p}{2}|\widehat{X}_s^{i,N}|^{p-2}\left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2\\&\quad+\frac{p(p-2)}{2}|\widehat{X}_s^{i,N}|^{p-4}\left|(\widehat{X}_s^{i,N})^\mathsf{T}\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2\Big|\mathcal{F}_{\underline{s}}\Big]\\&\le p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left(-\lambda|\widehat{X}_{\underline{s}}^{i,N}|^2+\left\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right\rangle+\frac{p-1}{2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})|^2\right)+\overline{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|\right),\end{aligned}\tag{47}$$

where $\overline{Q}_{p-2}(|\widehat{X}_{\underline{s}}^{i,N}|)$ is a polynomial in $|\widehat{X}_{\underline{s}}^{i,N}|$ of degree $p-2$.

We write

$$\begin{aligned}&\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p-|\widehat{X}_s^{i,N}|^p-p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right\rangle\\&=\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p-p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right\rangle\\&\quad+\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p-\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right|^p\right)-\left(|\widehat{X}_s^{i,N}|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p\right)\\&\quad-p\left(|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right\rangle-|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})z\right\rangle\right).\end{aligned}\tag{48}$$

Therefore, taking the conditional expectation on both sides of (48) and using Lemmas 4.8, 4.9, and 4.10, we obtain that for $\upsilon\ge\frac{1}{2\sqrt{N}}\max\{3L_3,1\}$,

$$\begin{aligned}&\mathbb{E}\left[\left|\widehat{X}_s^{i,N}+c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)z\right|^p-|\widehat{X}_s^{i,N}|^p-p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N}\right)z\right\rangle|\mathcal{F}_{\underline{s}}\right]\\&\le p\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N})\right|^2|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left(\frac{|z|^2}{2}+L_3^{-2}\left((1+|z|(L_3+\upsilon))^{p-1}-|z|(L_3+\upsilon)-1\right)\left(|z|\Big(\frac{L_3}{2}+\upsilon\Big)+\upsilon\right)\right)\\&\quad+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right)+\widehat{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|,|z|\right)\\&\quad+\sum_{j=1}^{p/2}\sum_{k=0}^{j}C_{j,k}|z|^{j+k}\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N},\delta_0)\Bigg(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-2}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-2}\\&+\sum_{\ell=2}^{j-k}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\right)\\&\quad+\sum_{\ell=2}^{p-2j}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\Bigg)+C_p|z|\left(\sum_{j=0}^{p-2}|\widehat{X}_{\underline{s}}^{i,N}|^j+\mathcal{W}_2(\mu_{\underline{s}}^{\widehat{\mathbf{X}}^N},\delta_0)\sum_{j=0}^{p-3}|\widehat{X}_{\underline{s}}^{i,N}|^j\right).\end{aligned}\tag{49}$$

Hence, combining (47) and (49), we get

$$\begin{aligned}
\mathbb{E}[\mathcal{R}_s|\mathcal{F}_{\underline{s}}] \le p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\Bigg(&- \lambda|\widehat{X}_{\underline{s}}^{i,N}|^2 + \langle \widehat{X}_{\underline{s}}^{i,N}, b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\rangle + \frac{p-1}{2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2 \\
&+ \left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2 \int_{\mathbb{R}_0^d} \Bigg(\frac{|z|^2}{2} + L_3^{-2}\Big((1+|z|(L_3+\upsilon))^{p-1} - |z|(L_3+\upsilon) - 1\Big) \\
&\times \Big(|z|\Big(\frac{L_3}{2}+\upsilon\Big)+\upsilon\Big)\Bigg)\nu(dz)\Bigg) + \overline{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|\right) \\
&+ \int_{\mathbb{R}_0^d}\Bigg(\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2, 2k-\ell, |\widehat{X}_{\underline{s}}^{i,N}|, 1+\widehat{U}_{\underline{s}}^{i,N}\right) + \widehat{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|, |z|\right) \\
&+ \sum_{j=1}^{p/2}\sum_{k=0}^{j} C_{j,k}|z|^{j+k}\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\Bigg(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-2} + \left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-2} \\
&+ \sum_{\ell=2}^{j-k}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell} + \left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\right) + \sum_{\ell=2}^{p-2j}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\Bigg) \\
&+ C_p|z|\Bigg(\sum_{j=0}^{p-2}|\widehat{X}_{\underline{s}}^{i,N}|^j + \mathcal{W}_2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\sum_{j=0}^{p-3}|\widehat{X}_{\underline{s}}^{i,N}|^j\Bigg)\Bigg)\nu(dz).
\end{aligned}$$

In the following, we choose $\lambda = \widetilde{\gamma}_1 + \frac{\widetilde{\gamma}_2}{N}$. It follows from Proposition 4.11 and the equality $\mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0) = \frac{1}{N}\sum_{m=1}^N |\widehat{X}_{\underline{s}}^{m,N}|^2$ that

$$\begin{aligned}
&- (\widetilde{\gamma}_1 + \frac{\widetilde{\gamma}_2}{N})|\widehat{X}_{\underline{s}}^{i,N}|^2 + \langle \widehat{X}_{\underline{s}}^{i,N}, b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\rangle + \frac{p-1}{2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2 \\
&\quad + \left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2 \int_{\mathbb{R}_0^d} \Bigg(\frac{|z|^2}{2} + L_3^{-2}\Big((1+|z|(L_3+\upsilon))^{p-1} - |z|(L_3+\upsilon) - 1\Big) \\
&\quad \times \Big(|z|\Big(\frac{L_3}{2}+\upsilon\Big)+\upsilon\Big)\Bigg)\nu(dz)\Bigg) \\
&\le - (\widetilde{\gamma}_1 + \frac{\widetilde{\gamma}_2}{N})|\widehat{X}_{\underline{s}}^{i,N}|^2 + \widetilde{\gamma}_1|\widehat{X}_{\underline{s}}^{i,N}|^2 + \widetilde{\gamma}_2\mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0) + \widetilde{\eta} = \frac{\widetilde{\gamma}_2}{N}\sum_{m=1, m\neq i}^{N}|\widehat{X}_{\underline{s}}^{m,N}|^2 + \widetilde{\eta}.
\end{aligned}$$

Therefore, using the estimate $\mathcal{W}_2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0) \le \frac{1}{\sqrt{N}}\sum_{m=1}^N |\hat{X}_{\underline{s}}^{m,N}|$, we obtain that

$$\begin{aligned}
\mathbb{E}[\mathcal{R}_s] \le p\mathbb{E}\Bigg[&\frac{\widetilde{\gamma}_2}{N}|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\sum_{m=1, m\neq i}^{N}|\widehat{X}_{\underline{s}}^{m,N}|^2 \\
&+ \int_{\mathbb{R}_0^d}\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2, 2k-\ell, |\widehat{X}_{\underline{s}}^{i,N}|, 1+\widehat{U}_{\underline{s}}^{i,N}\right)\nu(dz)\Bigg] \\
&+ C\sum_{j=0}^{p-2}\mathbb{E}\left[|\widehat{X}_{\underline{s}}^{i,N}|^j\right], \qquad (50)
\end{aligned}$$

for some positive constant C.

Thanks to Lemma 4.4, $\mathbb{E}[\mathcal{M}_t] = 0$. Now, we are going to take the expectation for (46) with $\lambda = \widetilde{\gamma}_1 + \frac{\widetilde{\gamma}_2}{N}$, plug (50) into (46), and use the inductive assumption, Condition **A4** and the following fact thanks to the independence between $|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}$ and $\sum_{m=1,m\neq i}^{N} |\widehat{X}^{m,N}_{\underline{s}}|^2$

$$\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\sum_{m=1,m\neq i}^{N}|\widehat{X}^{m,N}_{\underline{s}}|^2\right] = \sum_{m=1,m\neq i}^{N}\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\right]\mathbb{E}\left[|\widehat{X}^{m,N}_{\underline{s}}|^2\right],$$

$$\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-\ell}\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^{\ell}\right] = \mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-\ell}\right]\mathbb{E}\left[\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^{\ell}\right],$$

for any $\ell \in \{2,\dots,p-2\}$, where recall that $\widehat{U}^{i,N}_{\underline{s}} = \frac{1}{\sqrt{N}}\sum_{m=1;m\neq i}^{N}|\widehat{X}^{m,N}_{\underline{s}}|$. As a consequence, we get that

$$\mathbb{E}\left[e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})t}\left|\widehat{X}^{i,N}_t\right|^p\right] \leq |x_0|^p + C\int_0^t e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})s}\left(\sum_{\ell=2}^{p-2}\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-\ell}\right]\mathbb{E}\left[\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^{\ell}\right] + \sum_{j=0}^{p-2}\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{j}\right]\right)ds.$$

Here, recall that $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2$.

<u>Case $\widetilde{\gamma} > 0$:</u>

$$\begin{aligned}\mathbb{E}\left[e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})t}\left|\widehat{X}^{i,N}_t\right|^p\right] &\leq |x_0|^p + C\int_0^t e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})s}\left(\sum_{\ell=2}^{p-2}e^{(p-\ell)(\widetilde{\gamma}_1+\widetilde{\gamma}_2)s}e^{\ell(\widetilde{\gamma}_1+\widetilde{\gamma}_2)s} + \sum_{j=0}^{p-2}e^{j(\widetilde{\gamma}_1+\widetilde{\gamma}_2)s}\right)ds\\ &\leq |x_0|^p + \frac{C}{-p\widetilde{\gamma}_2(\frac{1}{N}-1)}\left(e^{-p\widetilde{\gamma}_2(\frac{1}{N}-1)t}-1\right),\end{aligned}$$

which implies that

$$\mathbb{E}\left[\left|\widehat{X}^{i,N}_t\right|^p\right] \leq |x_0|^p e^{p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})t} + \frac{C}{p\widetilde{\gamma}_2(1-\frac{1}{N})}\left(e^{p(\widetilde{\gamma}_1+\widetilde{\gamma}_2)t} - e^{p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})t}\right) \leq Ce^{p(\widetilde{\gamma}_1+\widetilde{\gamma}_2)t} = Ce^{p\widetilde{\gamma}t}.$$

<u>Case $\widetilde{\gamma} = 0$:</u> In this case, we have $\widetilde{\gamma}_1 + \frac{\widetilde{\gamma}_2}{N} < \widetilde{\gamma} = 0$. Using the integration by parts formula repeatedly, it can be checked that for any $\beta < 0$ and $q \in \mathbb{N}^*$,

$$\int_0^t e^{-\beta s}(1+s)^q ds \leq C(\beta,q)\left(e^{-\beta t}(1+t)^q + \int_0^t e^{-\beta s}ds\right) \leq C(\beta,q)e^{-\beta t}(1+t)^q,$$

for some constant $C(\beta,q) > 0$. This deduces that

$$\begin{aligned}\mathbb{E}\left[e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})t}\left|\widehat{X}^{i,N}_t\right|^p\right] &\leq |x_0|^p + C\int_0^t e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})s}\left(\sum_{\ell=2}^{p-2}(1+s)^{(p-\ell)/2}(1+s)^{\ell/2} + \sum_{j=0}^{p-2}(1+s)^{j/2}\right)ds\\ &\leq |x_0|^p + Ce^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})t}(1+t)^{p/2}.\end{aligned}$$

Therefore, we obtain that $\mathbb{E}\left[\left|\widehat{X}^{i,N}_t\right|^p\right] \leq C(1+t)^{p/2}$.

Case $\widetilde{\gamma} < 0$: In this case, we have $\widetilde{\gamma}_1 + \frac{\widetilde{\gamma}_2}{N} < \widetilde{\gamma} < 0$. Thus, we get $\mathbb{E}\left[e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})t}\left|\widehat{X}_t^{i,N}\right|^p\right] \leq |x_0|^p + C\int_0^t e^{-p(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})s}ds$, which implies that $\mathbb{E}\left[\left|\widehat{X}_t^{i,N}\right|^p\right] \leq C$. Consequently, combining with (41), we conclude that (45) holds for $k = k_0 + 1$. Thus, the result follows. $\square$

Remark 4.14. Thanks to Proposition 4.13, we get the following estimate on the expectation of the number of timesteps N_T

$$\mathbb{E}\left[N_T - 1\right] \leq \frac{C}{\Delta}, \tag{51}$$

for any $T > 0$, where the positive constant C does not depend on Δ.

Indeed, the same argument in the proof of Lemma 2 in [10] yields that $N_T = \sum_{k=1}^{N_T} 1 \leq \int_0^T \frac{1}{\Delta \mathbf{h}(\widehat{\boldsymbol{X}}_{\underline{t}}^N, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})}dt + 1$. Then, using (20), Assumption **A5**, and Remark 2.1 (i) and (iii), we get that for any $i \in \{1,\dots,N\}$ and $p_0 \geq 2(\ell+1)$,

$$\frac{1}{h(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})} \leq C\left(1 + |\widehat{X}_{\underline{t}}^{i,N}|^{p_0} + \mathcal{W}_2^{p_0}(\mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\right) \leq C\left(1 + |\widehat{X}_{\underline{t}}^{i,N}|^{p_0} + \frac{1}{N}\sum_{m=1}^N \left|\widehat{X}_{\underline{t}}^{m,N}\right|^{p_0}\right).$$

This, combined with $\mathbf{h}(\widehat{\boldsymbol{X}}_{\underline{t}}^N, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}) = \min\{h(\widehat{X}_{\underline{t}}^{1,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}), \dots, h(\widehat{X}_{\underline{t}}^{N,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})\}$, Lemma 4.4 and Proposition 4.13, shows the estimate (51).

Next, the difference between $\widehat{X}_t^{i,N}$ and $\widehat{X}_{\underline{t}}^{i,N}$ has the following uniform bound in time.

Lemma 4.15. *Let all conditions of Proposition 4.13 be satisfied. Then for any $p \in [2, p_0]$, there exists a positive constant $C_p = C(p, L)$ such that*

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[\left|\widehat{X}_t^{i,N} - \widehat{X}_{\underline{t}}^{i,N}\right|^p \middle| \mathcal{F}_{\underline{t}}\right] \leq C_p\Delta,$$

for any $t \geq 0$, and

$$\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|\widehat{X}_t^{i,N} - \widehat{X}_{\underline{t}}^{i,N}\right|^p\right] \leq C_p\Delta.$$

Proof. From (31), we have that for any $p \geq 2$,

$$|\widehat{X}_t^{i,N} - \widehat{X}_{\underline{t}}^{i,N}|^p \leq 3^{p-1}$$
$$\left[\left|b(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})\right|^p (t-\underline{t})^p + \left|\sigma_\Delta(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})\right|^p \left|W_t^i - W_{\underline{t}}^i\right|^p + \left|c_\Delta(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})\right|^p \left|Z_t^i - Z_{\underline{t}}^i\right|^p\right].$$

This, combined with (42), concludes the desired result. $\square$

4.3. Convergence

Theorem 4.16. *Let $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\eta}$, and υ be defined as in Proposition 4.11. Assume that $p_0 \geq 4\ell + 6$, $N \geq \left(\frac{\max\{3L_3,1\}}{2\upsilon}\right)^2$, and there exists a constant $\varepsilon > 0$ such that **A2** holds for $\kappa_1 = \kappa_2 = 1 + \varepsilon$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$. Then for any $T > 0$, there exist positive constants $C_T = C(x_0, L, L_1, L_2, L_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \varepsilon, T)$ and $C'_T = C'(x_0, L, L_1, L_2, L_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \varepsilon, T)$ such that*

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^2\right] \le C_T\Delta, \tag{52}$$

and for any $p \in (0,2)$,

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^p\right] \le \left(\frac{4-p}{2-p}\right)(C_T'\Delta)^{p/2}. \tag{53}$$

Moreover, if $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$, and $L_1 + L_2 < 0$, then, there exists a positive constant $C'' = C''(x_0, L, L_1, L_2, L_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \varepsilon)$, which does not depend on T, such that

$$\max_{i\in\{1,\dots,N\}} \sup_{t\ge 0} \mathbb{E}\left[\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^2\right] \le C''\Delta. \tag{54}$$

Proof. Thanks to (14), (32), and Itô's formula, we obtain that for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
e^{-\lambda t}|X_t^{i,N} - \widehat{X}_t^{i,N}|^2 &= \int_0^t e^{-\lambda s}\Big(-\lambda|X_s^{i,N} - \widehat{X}_s^{i,N}|^2 + 2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle \\
&+ \left|\sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\Big)ds + \mathcal{M}_t \\
&+ \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\left|\left(c(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right)z\right|^2 \nu(dz)ds,
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
\mathcal{M}_t &= 2\int_0^t e^{-\lambda s}\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, \left(\sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - \sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right)dW_s^i\right\rangle \\
&\quad + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\left(|X_{s-}^{i,N} - \widehat{X}_{s-}^{i,N} + \left(c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}) - c_\Delta(\widehat{X}_{\underline{s}-}^{i,N}, \mu_{\underline{s}-}^{\widehat{\boldsymbol{X}}^N})\right)z|^2 - |X_{s-}^{i,N} - \widehat{X}_{s-}^{i,N}|^2\right)\widetilde{N}^i(ds, dz).
\end{aligned}$$

In the following, we will give upper bounds for each term on the right-hand side of (55). First, we decompose

$$2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle = 2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - b\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle + S, \tag{56}$$

where $S = 2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle$. By using Cauchy's inequality and Condition **A3**, we obtain that for any $\varepsilon_1 > 0$,

$$\begin{aligned}
S &\le \varepsilon_1\left|X_s^{i,N} - \widehat{X}_s^{i,N}\right|^2 + \frac{1}{\varepsilon_1}\left|b\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2 \\
&\le \varepsilon_1\left|X_s^{i,N} - \widehat{X}_s^{i,N}\right|^2 + \frac{6}{\varepsilon_1}L^2\left(1 + \left|\widehat{X}_s^{i,N}\right|^{2\ell} + \left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\right)\left(\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 + \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right) \\
&\le \varepsilon_1\left|X_s^{i,N} - \widehat{X}_s^{i,N}\right|^2 + \frac{6}{\varepsilon_1}L^2\Bigg(\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 + \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right) + 2^{2\ell-1}\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell+2}\Bigg)
\end{aligned}$$

$$+\frac{6}{\varepsilon_1}L^2\left(2^{2\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{2\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2\right)$$
$$+\frac{6}{\varepsilon_1}L^2(2^{2\ell}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right). \tag{57}$$

Second, by using Cauchy's inequality we have that for any $\varepsilon_2>0$,

$$\begin{aligned}&\left|\sigma(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\\
&\leq(1+\varepsilon_2)\left|\sigma(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-\sigma\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\\
&\quad+2\left(1+\frac{1}{\varepsilon_2}\right)\left(\left|\sigma\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)-\sigma\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2+\left|\sigma\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)-\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\right).\end{aligned}\tag{58}$$

It follows from Remark 2.1 that

$$\begin{aligned}&\left|\sigma\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)-\sigma\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\\
&\leq\frac{L\widetilde{L}}{1+\varepsilon}\left(1+\left|\widehat{X}_s^{i,N}\right|^\ell+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\right)\left(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right)\\
&\leq\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\\
&\quad+(2^\ell+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+(2^{\ell-1}+1)|\widehat{X}_{\underline{s}}^{i,N}|^\ell\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg).\end{aligned}\tag{59}$$

Then, it follows from **T5** and Remark 2.1(iii) that

$$\begin{aligned}&\left|\sigma\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)-\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\leq\Delta\left|\sigma\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^4\left(1+\left|\widehat{X}_{\underline{s}}^{i,N}\right|\right)^2\\
&\leq4\Delta\left(16\left(\frac{L\widetilde{L}}{1+\varepsilon}\right)^2\left(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\right)\left(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\right)+4\left|\sigma(0,\delta_0)\right|^4\right)\left(1+|\widehat{X}_{\underline{s}}^{i,N}|^2\right).\end{aligned}\tag{60}$$

Third, using Cauchy's inequality, we obtain that for any $\varepsilon_3>0$,

$$\begin{aligned}&\left|c(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\leq(1+\varepsilon_3)\left|c(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\\
&\quad+2\left(1+\frac{1}{\varepsilon_3}\right)\left(\left|c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)-c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2+\left|c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)-c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\right).\end{aligned}\tag{61}$$

Thanks to Remark 2.1(ii), we have

$$\begin{aligned}&\left|c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)-c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\int\limits_{\mathbb{R}_0^d}|z|^2\nu(dz)\\
&\leq\frac{L\widetilde{L}}{1+\varepsilon}\left(1+\left|\widehat{X}_s^{i,N}\right|^\ell+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\right)\left(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right)\\
&\leq\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\\
&\quad+(2^\ell+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+(2^\ell+1)|\widehat{X}_{\underline{s}}^{i,N}|^\ell\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg).\end{aligned}\tag{62}$$

Thanks to **T5** and Remark 2.1(iii) and (i), we have

$$
\begin{aligned}
&\left|c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)-c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\\
\le&\Delta\left|c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^4\left(1+\left|\widehat{X}_{\underline{s}}^{i,N}\right|+\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|\right)^2\\
\le&\frac{6\Delta}{\left(\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\right)^2}\left(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\Big)\Big(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|c(0,\delta_0)|^4\Big(\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\Big)^2\right)\\
&\times\left(1+|\widehat{X}_{\underline{s}}^{i,N}|^2+2\left(|b(0,\delta_0)|^2+4L^2\big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\big)\left(|\widehat{X}_{\underline{s}}^{i,N}|^2+\mathcal{W}_2^2\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\right)\right)\right).
\end{aligned}
\tag{63}
$$

Therefore, inserting all estimations (56) – (63) into (55), and choosing $\varepsilon_2=\varepsilon_3=\varepsilon$, we obtain that for any $\lambda\in\mathbb{R}$ and $\varepsilon_1>0$,

$$
\begin{aligned}
&e^{-\lambda t}|X_t^{i,N}-\widehat{X}_t^{i,N}|^2\le\mathcal{M}_t+\\
&+\int_0^t e^{-\lambda s}\Bigg(-\lambda|X_s^{i,N}-\widehat{X}_s^{i,N}|^2+\varepsilon_1\left|X_s^{i,N}-\widehat{X}_s^{i,N}\right|^2+2\left\langle X_s^{i,N}-\widehat{X}_s^{i,N},b(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-b\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle\\
&+(1+\varepsilon)\left|\sigma(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-\sigma\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right|^2+(1+\varepsilon)\Big|c(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})\\
&-c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\Big|^2\int_{\mathbb{R}_0^d}|z|^2\nu(dz)+\mathcal{R}_s\Bigg)ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R}_s=&\frac{6}{\varepsilon_1}L^2\Bigg[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{2\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell+2}+2^{2\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\\
&\times\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{2\ell}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+(2^{2\ell}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg]\\
&+2(1+\frac{1}{\varepsilon})\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\\
&\times\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^\ell+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+(2^\ell+1)|\widehat{X}_{\underline{s}}^{i,N}|^\ell\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg)\\
&+4\Delta\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\Big)\Big(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|\sigma(0,\delta_0)|^4\Bigg)\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^2\Big)\Bigg]\\
&+2\left(1+\frac{1}{\varepsilon}\right)\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}+2^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\\
&\times\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^\ell+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+(2^\ell+1)|\widehat{X}_{\underline{s}}^{i,N}|^\ell\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg)\\
&+\frac{6\Delta}{\int_{\mathbb{R}_0^d}|z|^2\nu(dz)}\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\Big)\Big(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|c(0,\delta_0)|^4\Big(\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\Big)^2\Bigg)
\end{aligned}
$$

$$\times\left(1+|\widehat{X}_{\underline{s}}^{i,N}|^2+2\left(|b(0,\delta_0)|^2+4L^2\left(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\right)\left(|\widehat{X}_{\underline{s}}^{i,N}|^2+\mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\right)\right)\right)\Bigg].$$

Using Condition **A2** for $\kappa_1=\kappa_2=1+\varepsilon$, $L_1\in\mathbb{R}$, $L_2\geq 0$, we obtain that for any $\lambda\in\mathbb{R}$ and $\varepsilon_1>0$,

$$\begin{aligned}
&e^{-\lambda t}|X_t^{i,N}-\widehat{X}_t^{i,N}|^2\leq \mathcal{M}_t+\\
&+\int_0^t e^{-\lambda s}\Bigg\{-\lambda|X_s^{i,N}-\widehat{X}_s^{i,N}|^2+\varepsilon_1\left|X_s^{i,N}-\widehat{X}_s^{i,N}\right|^2+L_1|X_s^{i,N}-\widehat{X}_s^{i,N}|^2\\
&+L_2\mathcal{W}_2^2\left(\mu_s^{\boldsymbol{X}^N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)+\mathcal{R}_s\Bigg\}ds.
\end{aligned}\tag{64}$$

Now, using Lemma 4.15 and Proposition 4.13, we have the following estimates

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{W}_2^2(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right]\leq\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\left|\widehat{X}_s^{j,N}-\widehat{X}_{\underline{s}}^{j,N}\right|^2\right]=\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2\right]\leq C\Delta,\\
&\mathbb{E}\left[\mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\right]=\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N\left|\widehat{X}_{\underline{s}}^{j,N}\right|^2\right]=\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{j,N}\right|^2\right]=\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^2\right]\leq C,\\
&\mathbb{E}\left[\mathcal{W}_2^4(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\right]=\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^N\left|\widehat{X}_{\underline{s}}^{j,N}\right|^2\right)^2\right]\leq\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{j,N}\right|^4\right]=\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^4\right]\leq C,
\end{aligned}\tag{65}$$

for any $i\in\{1,\dots,N\}$ and some constant $C>0$.

Using the estimate $\mathcal{W}_2^p\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\leq\frac{1}{N}\sum_{j=1}^N\left|\widehat{X}_s^{j,N}-\widehat{X}_{\underline{s}}^{j,N}\right|^p$, valid for any $p\geq 2$ and Lemma 4.15, we have that for $\rho\in\{\ell,2\ell\}$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\rho\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right]\leq\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\rho\left|\widehat{X}_s^{j,N}-\widehat{X}_{\underline{s}}^{j,N}\right|^2\right]\\
&=\frac{1}{N}\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\rho+2}\right]+\frac{1}{N}\sum_{j\neq i}\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\rho\right]\mathbb{E}\left[\left|\widehat{X}_s^{j,N}-\widehat{X}_{\underline{s}}^{j,N}\right|^2\right]\leq C\Delta.
\end{aligned}\tag{66}$$

Next, using Lemma 4.15, Proposition 4.13 and the fact that $p_0\geq 4\ell+6$, we get that

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^q\right]\leq C\Delta,\ \ \mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\right]\leq\left(\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\right]\right)^{1/2}\leq C\sqrt{\Delta}\leq C,\\
&\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\iota\right]\leq C,
\end{aligned}\tag{67}$$

for $q\in\{2;\ell+2;2\ell;2\ell+2\}$, $\iota\in\{\ell;2\ell;2\ell+4;2\ell+6;4\ell+6\}$, and

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2\right]=\mathbb{E}\left[\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2|\mathcal{F}_{\underline{s}}\right]\right]\\
&=\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa\mathbb{E}\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2|\mathcal{F}_{\underline{s}}\right]\right]\leq C\Delta\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa\right]\leq C\Delta,
\end{aligned}\tag{68}$$

for $\kappa\in\{\ell;2\ell\}$ and some constant $C>0$.

Furthermore, using Lemma 4.15 and Proposition 4.13, we obtain that for $\varrho \in \{\ell, 2\ell\}$,

$$\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\varrho} \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right] \leq \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\varrho}\left|\widehat{X}_s^{j,N} - \widehat{X}_{\underline{s}}^{j,N}\right|^2\right]$$

$$= \frac{1}{N}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\varrho}\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2\right] + \frac{1}{N}\sum_{j\neq i}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\varrho}\right]\mathbb{E}\left[\left|\widehat{X}_s^{j,N} - \widehat{X}_{\underline{s}}^{j,N}\right|^2\right] \leq C\Delta. \tag{69}$$

Using Proposition 4.13, we obtain that for $\vartheta \in \{2\ell + 2; 4\ell + 2\}$,

$$\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\vartheta} \mathcal{W}_2^4(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\right] \leq \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\vartheta}\left|\widehat{X}_{\underline{s}}^{j,N}\right|^4\right]$$

$$= \frac{1}{N}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\vartheta+4}\right] + \frac{1}{N}\sum_{j\neq i}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\vartheta}\right]\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{j,N}\right|^4\right] \leq C. \tag{70}$$

Proceeding similarly to the above, we get that

$$\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{4\ell} \mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\mathcal{W}_2^4(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\right] = \mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{4\ell} \mathcal{W}_2^6(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\right] \leq C. \tag{71}$$

Consequently, plugging (65)-(71) into (64), taking the expectation on both sides and choosing $\lambda = \varepsilon_1 + L_1 + L_2$, we get that for any $t \in [0, T]$,

$$\mathbb{E}\left[e^{-(\varepsilon_1+L_1+L_2)t}\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^2\right] \leq C\Delta\int_0^t e^{-(\varepsilon_1+L_1+L_2)s}ds.$$

This implies that

$$\max_{i\in\{1,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\left[|X_t^{i,N} - \widehat{X}_t^{i,N}|^2\right] \leq C_T\Delta, \tag{72}$$

for some positive constant $C_T = C(x_0, L, L_1, L_2, L_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \varepsilon, T)$, which shows (52).

Next, for any stopping time $\tau \leq T$, using again (64) with $t = \tau$, $\lambda = \varepsilon_1 + L_1$, taking the expectation of the above inequality on both sides and using again the estimates (65)–(71) and (72), we obtain that

$$\mathbb{E}\left[e^{-(\varepsilon_1+L_1)\tau}|X_\tau^{i,N} - \widehat{X}_\tau^{i,N}|^2\right] \leq \int_0^T e^{-(\varepsilon_1+L_1)s}\mathbb{E}\left[L_2\mathcal{W}_2^2\left(\mu_s^{\boldsymbol{X}^N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right) + \mathcal{R}_s\right]ds \leq \widetilde{C}_T\Delta,$$

for some constant $\widetilde{C}_T > 0$. Therefore, due to Proposition IV.4.7 in [30], we get that for any $p \in (0, 2)$,

$$\max_{i\in\{1,\ldots,N\}}\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\frac{p(\varepsilon_1+L_1)t}{2}}|X_t^{i,N} - \widehat{X}_t^{i,N}|^p\right] \leq \left(\frac{2-p/2}{1-p/2}\right)(\widetilde{C}_T\Delta)^{p/2},$$

which, combined with the fact that $e^{-\frac{p(\varepsilon_1+L_1)t}{2}} \geq e^{-\frac{p|\varepsilon_1+L_1|T}{2}}$, concludes (53).

Moreover, when $L_1 + L_2 < 0$, we can always choose $\varepsilon_1 > 0$ such that $L_1 + L_2 + \varepsilon_1 < 0$. Consequently, when $L_1 + L_2 < 0$ and $\widetilde{\gamma} < 0$, the constant C_T in (72) now does not depend on T. Therefore, we have shown (54), which finishes the desired proof. □

We now state our main result on strong convergence in both finite and infinite time intervals of the tamed-adaptive Euler-Maruyama scheme for multidimensional McKean-Vlasov SDEs driven by Lévy processes.

Theorem 4.17. *Let $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\eta}$, and υ be defined as in Proposition 4.11. Assume that $p_0 \geq 4\ell + 6$, $N \geq \left(\frac{\max\{3L_3,1\}}{2\upsilon}\right)^2$, and there exists a constant $\varepsilon > 0$ such that **A2** holds for $\kappa_1 = \kappa_2 = 1 + \varepsilon$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$. Then for any $T > 0$, there exists a positive constant $C_T = C(x_0, L, L_1, L_2, L_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \varepsilon, T)$ such that*

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^i - \widehat{X}_t^{i,N}\right|^2\right] \leq C_T\left(\Delta + \varphi(N)\right), \tag{73}$$

for any $N \in \mathbb{N}$, where the constant $C_T > 0$ does not depend on N.

Moreover, if $\gamma = \gamma_1 + \gamma_2 < 0$, $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$ and $L_1 + L_2 < 0$, then there exists a positive constant $C'' = C''(x_0, L, L_1, L_2, L_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \varepsilon)$ which does not depend on T such that

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^i - \widehat{X}_t^{i,N}\right|^2\right] \leq C''\left(\Delta + \varphi(N)\right). \tag{74}$$

Proof. As a consequence of Proposition 3.2 and Theorem 4.16, the proof is straightforward. Thus, we omit it. □

5. Numerical experiment

In this section, we consider the rate of convergence of the tamed-adaptive Euler-Maruyama scheme (19), (20) and (21) in Theorem 4.16 for a fixed large value of N. We will consider two models. The first model is the Ginzburg-Landau equation with jumps and mean-field, in which both $L_1 + L_2$ and $\tilde{\gamma}$ are positive. The second model is a particular SDE with jumps and mean-field, in which both $L_1 + L_2$ and $\tilde{\gamma}$ are negative. Let $Z = (Z_t)_{t\geq 0}$ be a bilateral Gamma process whose scale parameter is γ_Z and whose shape parameter is λ_Z.

Model 1:

$$dX_t = 0.1\left(X_t - X_t^3 + \mathbb{E}\left[X_t\right]\right)dt + 0.1X_t dW_t + \sin(X_{t-})dZ_t. \tag{75}$$

Model 2:

$$dX_t = \left(-1 - 3\left(X_t + \mathbb{E}\left[X_t\right]\right) - X_t|X_t|^{0.3}\right)dt + 0.2\left(1 + |X_t|^{1.1} + \mathbb{E}\left[X_t\right]\right)dW_t + 0.2\left(X_{t-} + \mathbb{E}\left[X_{t-}\right]\right)dZ_t. \tag{76}$$

Verifying that these coefficients satisfy Conditions **A1**–**A6** is straightforward. We implement the tamed-adaptive Euler approximation scheme (19)–(21) with $N = 500$, $X_0 = 1, \ell = 2, p_0 = 12, \gamma_Z = \lambda_Z = 1$; $T = 1$, $T = 5$, $T = 10$ for Model 1; and $N = 500$, $X_0 = 1, \ell = 0.3, p_0 = 8, \gamma_Z = 1, \lambda_Z = 5$; and $T = 1$, $T = 5$, $T = 10$ for Model 2.

Since the exact solutions of equations (75) and (76) are unknown, we will derive the rate of convergence of the tamed-adaptive Euler approximation scheme (19)–(21) in an indirect way as in [16,17]. We consider the mean squared difference of $\widehat{X}$ on two consecutive levels as follows: $\mathrm{MSE}(\mathbf{l}, T) = \frac{1}{M}\sum_{k=1}^{M}|\widehat{X}_T^{(\mathbf{l},k)} - \widehat{X}_T^{(\mathbf{l}+1,k)}|^2$, where for each $\mathbf{l} \geq 1$, $(\widehat{X}^{(\mathbf{l},k)})_{1\leq k\leq M}$ is a sequence of independent copies of $\widehat{X}^{(\mathbf{l})}$ defined by equations (19)–(21) with $\Delta = 2^{-\mathbf{l}}$. Here $\widehat{X}_T^{(\mathbf{l},k)}$ and $\widehat{X}_T^{(\mathbf{l}+1,k)}$ must be simulated to the same Brownian motion and bilateral Gamma process (See Algorithm 1 in [10]). It is clear that $\widehat{X}^{(\mathbf{l})}$ converges at some rate of order $\beta \in (0, +\infty)$ in L^2-norm iff $2^{\beta\mathbf{l}}\|\widehat{X}_T^{(\mathbf{l}+1)} - \widehat{X}_T^{(\mathbf{l})}\|_{L^2} = O(1)$, which implies that $\log_2 \mathrm{MSE}(\mathbf{l}, T) = -2\beta\mathbf{l} + C + o(1)$, for some constant $C \in \mathbb{R}$. Thus we can use the regression method to estimate the rate β.

Fig. 1 presents the values of $\log_2 \mathrm{MSE}(\mathbf{l}, T)$ plotted against $\mathbf{l} \in \{1, 2, \ldots, 6\}$ for Model 1 (left panel) and

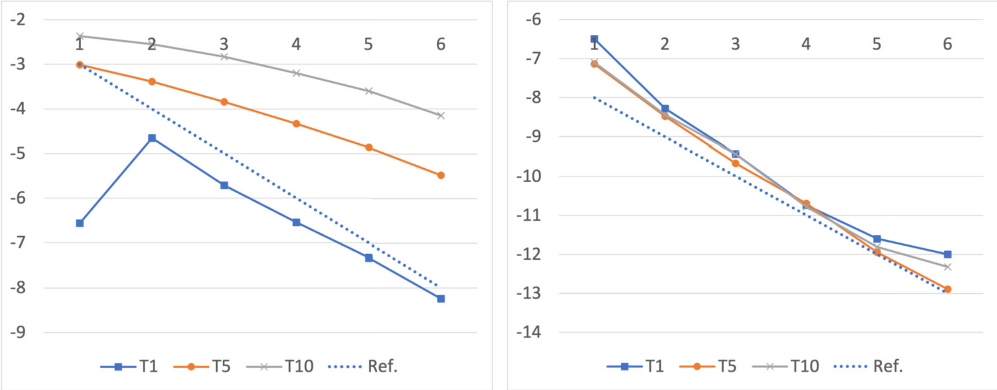

Fig. 1. Error $\log_2 \mathrm{MSE}(\mathbf{l}, 10)$ plotted against $\mathbf{l} = 1, \ldots, 6$ for Model 1 (left) and Model 2 (right). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Model 2 (right panel). Each panel comprises three graphs corresponding to $T = 1$ (T1), $T = 5$ (T5), and $T = 10$ (T10), along with a dashed reference line with a slope of -1.

For Model 1, we observe that the TAEM scheme converges at a rate of order $\beta \approx 1/2$ for $T = 1$. However, as T increases, the convergence rate of the TAEM scheme decreases and the mean square error (MSE) increases. In contrast, for Model 2, the TAEM scheme converges at a rate of order $\beta \approx 1/2$ for any value of T, and the mean square error (MSE) remains independent of T.

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