A NEW APPROACH TO APPROXIMATE SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS *

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Abstract

In this paper, we derive a new approach to approximate solutions of stochastic differential equations in point of view of stochastic analysis on time scales. We will prove that the solutions of stochastic dynamic equations on time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ converges in the mean-square to the solution of stochastic dynamic equations on time scale $\mathbb{T}$ provided $\mathbb{T}_n$ tends to $\mathbb{T}$. Further, the optimal rate of convergence can be recovered if the drift and diffusion coefficients are Lipchitz in both variables t and x. This work can be considered as a generalization of the Euler-Maruyama approximation for solutions of stochastic differential equations.

1 Introduction

Stochastic differential equations (SDEs) play an important role of investigation in many scientific disciplines, for instance in physics, chemistry, biology, and economics, etc. Therefore, finding solutions of SDEs is important on both in theory and practice. Unfortunately, for almost SDEs we can not find the explicit solutions. This leads to research numerical methods for solving its approximation solutions which has attracted a good deal of attention from researchers in recent years (see, e.g. [8, 9, 15, 16]).

In this paper we will study this on the general contex by using stochatisc analysis on time scales which is an nonempty closed subset of the real numbers (see, e.g. [3, 5, 19, 20]). Recently, the stochastic dynamic equations on time scales have studied in [6, 7, 17, 18]. The problem is considered as approximating the solution of stochastic dynamic equation on a time scale $\mathbb{T}$ by the solutions of stochastic dynamic equations on another time scales $\mathbb{T}_n$ which converge to $\mathbb{T}$. In the case of deterministic dynamic equations, we have studied the similar problem in [10, 11, 12].

Fix $a \in \mathbb{R}$. Denote by $\mathcal{T}(a)$ the set of all time scales $\mathbb{S}$ with $a \in \mathbb{S}$. We endow $\mathcal{T}$ with a Hausdorff distance

$$d_H(\mathbb{S}_1, \mathbb{S}_2) = \max\{\sup_{t_2\in\mathbb{S}_1}\inf_{t_1\in\mathbb{S}_2}|t_1 - t_2|; \sup_{t_1\in\mathbb{S}_1}\inf_{t_2\in\mathbb{S}_1}|t_1 - t_2|\}. \tag{1.1}$$

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Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant a}, \mathbb{P})$ be a probability space satisfying usual conditions. i.e., $\mathcal{F}$ is P complete and the filtration $\{\mathcal{F}_t\}_{t\geqslant a}$ is right continuous. Let M be a $\mathcal{F}_t$–square martingale defined on $[a, \infty)$.

For any time scale $\mathbb{S} \in \mathcal{T}(a)$, the restriction of M in $\mathbb{S}$ is a square martingale. Therefore, we can consider the stochastic dynamic equation of Itô type on time scale $\mathbb{S}$ (see [6])

$$\begin{cases} d^\nabla x(t) = f(t, x(t_-))d^\nabla t + g(t, x(t_-))d^\nabla M(t), \quad t \in [a, \infty)_{\mathbb{S}}, \\ x(a) = \xi, \end{cases} \tag{1.2}$$

where $f, g : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^s$ are two measurable functions.

The existence and uniqueness of solution for the equation (1.2) can be founded in [6].

When $\mathbb{S} = \mathbb{R}$, this equation becomes a stochastic differential equation and if $\mathbb{S} = h\mathbb{Z}$ then it is a stochastic difference equation.

The aim of this paper is to consider the convergence of solutions for stochastic dynamic equations on time scales. Precisely,

- Let $\{\mathbb{T}_n\}_{n\in\mathbb{N}} \subset \mathcal{T}(a)$ be a sequence of time scales. Suppose that $\{\mathbb{T}_n\}_{n\in\mathbb{N}}$ converges to the time scale $\mathbb{T}$ in Hausdorff distance d_H. Let $x_n(\cdot)$ (resp. $x(\cdot)$) be the solution of (1.2) with $\mathbb{S} = \mathbb{T}_n$ (resp. with $\mathbb{S} = \mathbb{T}$.
- Under which conditions, $x_n(t) \to x(t)$ when $n \to \infty$.
- Can we estimate the rate of convergence

There is two natural questions arise that:

-*The solution of equation* (**??**) *converges to the solution of equation* (1.2) *or not as* $H - \lim\limits_{n\to\infty} \mathbb{T}_n = \mathbb{T}$?

-*?*

The aim of this paper is to answer two above questions. In particular cases, $\mathbb{T} = [a, b]$, $T_n = \{t_i, i = \overline{0, n}\}$ is a partition of $[a, b]$ and $M(t)$ is a Brownian motion, we obtain the convergence of the Euler-Maruyama approximation which is a most well known method for SDEs (see [14]).

The organization of the paper is as follows. In Section 2, we survey some basic notions and properties of analysis on time scales. In Section 3, we study the convergence of solutions of stochastic dynamic equations on time scales. The main results of the paper are derived here. Section 4 is concerned with the convergence of the Euler-Maruyama method and some special examples to illustrate results. The last section deals with some conclusions.

2 Preliminaries

This section surveys some basic notions on the theory of the analysis on time scales introduced by Stefan Hilger 1988 [13].

A time scales is a nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$. We assume throughout the paper that a time scales $\mathbb{T}$ is endowed with the topology induced by the standard topology on $\mathbb{R}$. We define the *forward jump operator* and the *backward jump operator* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (supplemented by $\inf \emptyset = \sup \mathbb{T}$) and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (supplemented by $\sup \emptyset = \inf \mathbb{T}$). The *backward graininess* is given by $\nu(t) = t - \rho(t)$. A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is right-scattered and left-scattered. For every $a, b \in \mathbb{R}, a < b$, by $[a, b]_{\mathbb{T}}, (a, b]_{\mathbb{T}}, [a, b)_{\mathbb{T}}$, we mean the sets $\{t \in \mathbb{T} : a \leqslant t \leqslant b\}, \{t \in \mathbb{T} : a < t \leqslant b\}, \{t \in \mathbb{T} : a \leqslant t < b\}$, respectively. For $a \in \mathbb{T}$, denote $\mathbb{T}(a) = \{x \in \mathbb{T} : x \geqslant a\}$.

A function f defined on $\mathbb{T}$ is *ld-continuous* if it is continuous at every left-dense point and the right-sided limit exists at every right-dense point. Similarly, one has the notation of rd-continuous. A function f is called regulated if the right-sided limits exist at all right-dense points and the left-sided limits exist at all left-dense points. For regulated functions f defined on $\mathbb{T}$, denote $\lim_{\sigma(s)\uparrow t} f(s)$ by $f(t_-)$. It is easy to see that if t is left-scattered and f is a continuous function then $f(t_-) = f(\rho(t))$. Let $\mathbb{I} = \{t : t \text{ is left-scattered}\}$. Clearly, the set $\mathbb{I}$ of all left-scattered of $\mathbb{T}$ is at most countable.

Definition 2.1. *(Nabla Derivative) A function $f : \mathbb{T} \to \mathbb{R}^d$ is called nabla differentiable at t if there exists a vector $f^\nabla(t)$ such that for all $\epsilon > 0$*

$$\|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)\| \leqslant \epsilon|\rho(t) - s|$$

for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$ and for some $\delta > 0$. The vector $f^\nabla(t)$ is called the nabla derivative of f at t.

If $\mathbb{T} = \mathbb{R}$ then the nabla derivative is $f^{'}(t)$ from continuous calculus; if $\mathbb{T} = \mathbb{Z}$ then the nabla derivative is the backward difference, $\nabla f(t) = f(t) - f(t-1)$, from discrete calculus.

Let A be an increasing right continuous function defined on $\mathbb{T}$. Denote by $\mathcal{B}_1 = \{(a, b]_{\mathbb{T}} : a, b \in \mathbb{T}\}$ the family of all left open and right closed interval of $\mathbb{T}$. It is easy to see that $\mathcal{B}_1$ is semi-ring of subsets of $\mathbb{T}$. Let m_1 be the set function defined on B_1 by $m_1((a, b]_{\mathbb{T}}) = A(b) - A(a)$. It is easy to show that m_1 is a countably additive measure on $\mathcal{B}_1$. Then, we wite μ_∇^A for the Caratheodory extension of the set function m_1 associated with the family $\mathcal{B}_1$ and call it the Lebesgue-Stieltjes nabla measure associated with A on $\mathbb{T}$. Let $f : (a, b]_{\mathbb{T}} \to \mathbb{R}$ be an μ_∇^A-measurable function. Then, the integral of f associated with the measure μ_∇^A on $(a, b]$, denoted by $\int_a^b f(t)\nabla A(t)$ is called Lebesgue-Stieltjes nabla integral. If $A(t) = t$ for all $t \in \mathbb{T}$ then μ_∇^A is Lebesgue nabla measure and $\int_a^b f(t)\nabla t$ is Lebesgue nabla integral. For details, we can refer to [4].

Now, we fix a $M = \{M_t\}_{t\geqslant a} \in \mathcal{M}_2$ (the set of the square integrable martingales defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}}, \mathbb{P})$) with the characteristic $\langle M\rangle_t$ (see [5]). A function f defined on $(a, b]_{\mathbb{T}}$ is called simple if there exist a partition π of $[a, b]_{\mathbb{T}} : a = a < t_1 < \ldots < t_n = b$ and bounded random variables $\{f_i\}$ such that f_i is $\mathcal{F}_{t_{i-1}}$ measurable for all $i = 1, 2, \ldots n$ and

$$f(t) = \sum_{i=1}^{n} f_i \chi_{(t_{i-1}, t_i]}(t),\ t \in (a, b]_{\mathbb{T}}. \tag{2.1}$$

For a simple function f in (2.1), we define the stochastic nabla integral

$$\int_a^b f(t)\nabla M(t) = \sum_{i=1}^{n} f_i(M_{t_i} - M_{t_{i-1}}).$$

In general, let f be a real-valued predictable function such that $\int_a^b |f(t)|^2 \nabla \langle M\rangle_t < \infty$. Then, there exists a sequence of simple functions $\{f_n\}$ such that

$$\lim_{n\to\infty} \mathbb{E} \int_a^b |f(t) - f_n(t)|^2 \nabla \langle M\rangle_t = 0.$$

The stochastic nabla integral will be defined by

$$\int_a^b f(t)\nabla M(t) = \lim_{n\to\infty} \int_a^b f_n(t)\nabla M(t). \tag{2.2}$$

This integral is independent on choosing of sequences $\{f_n\}$. For details, we refer interested readers to [5].

Let p be a regulated function defined on $\mathbb{T}$. Then the exponential function $e_p(\mathbb{T}; t, a)$ defined by [2, Definition 1.38, pp. 10] is a solution of the initial value problem

$$y^{\nabla}(t) = p(t_-)y(t_-), \;\; y(a) = 1, \; t \in \mathbb{T}(a),$$

(see [6]). It is easy to see that

$$y(t) = \exp\left\{\int_a^t \lim_{h\downarrow \nu(s)} \frac{\ln(1 + hp(s))}{h} \nabla s\right\}.$$

Since the function $\dfrac{\ln(1 + hp(s))}{h}$ is decreasing in $h \geqslant 0$ for any p to be positively regressive, it follows that when $\mathbb{T}_1 \subset \mathbb{T}_2$ we have

$$0 < e_p(\mathbb{T}_1; t, a) < e_p(\mathbb{T}_2, t, a), \; \forall t \in \mathbb{T}_1(a). \tag{2.3}$$

For other properties of exponential function $e_p(\mathbb{T}; t, s)$, the interested reader can see [1, 2]. The following lemma is a type of the Gronwall inequality which will be used in next section.

Lemma 2.2 ([6])**.** *Let $u(t)$ be a regulated function and $u_0, \alpha \in \mathbb{R}_+$. Then, the inequality*

$$u(t) \leqslant u_0 + \alpha \int_a^t u(s_-)\nabla_{\mathbb{T}} s \quad \textit{for all} \quad t \in \mathbb{T}(a),$$

implies

$$u(t) \leqslant u_0 e_\alpha(\mathbb{T}; t, a) \quad \textit{for all } t \in \mathbb{T}(a).$$

3 Main results

Let $\{\mathbb{T}; \mathbb{T}_n\}_{n\in\mathbb{N}} \subset \mathfrak{T}(a)$ be a sequence of time scales satisfying

$$H - \lim_{n\to\infty} \mathbb{T}_n = \mathbb{T}.$$

We define the time scale

$$\widehat{\mathbb{T}} = \overline{\bigcup_{n\in\mathbb{N}} \mathbb{T}_n \bigcup \mathbb{T}}. \tag{3.1}$$

Let M be a square integrable martingale defined on $\widehat{\mathbb{T}}$ and denote by $\langle M\rangle_t$ the quadratic variation of M. Suppose that $\langle M\rangle_t$ is absolutely continuous with respect to Lebesgue measure on $\widehat{\mathbb{T}}$. That is, there exists a progressively measurable process $N_t, t \geqslant 0$ such that

$$\langle M\rangle_t = \int_a^t N_\tau \nabla^{\widehat{\mathbb{T}}}\tau, \quad \forall \; t \in \widehat{\mathbb{T}}. \tag{3.2}$$

Further, assume that N bounded on $\widehat{\mathbb{T}}$ by a constant $\hbar$, i.e.,

$$\mathbb{P}\{\sup_{t\in\widehat{T}} N_t \leqslant \hbar\} = 1. \tag{3.3}$$

Denote by ρ_n (resp. σ_n) the backward jump (resp. forward jump) operator on the time scale $\mathbb{T}_n$. For any $t \in \mathbb{T}$, there exists a unique $s \in \mathbb{T}_n$, say $s = \gamma^{\mathbb{T},\mathbb{T}_n}(t)$, such that either $s = t$ or $t \in (\rho_n(s), s)$. It is easy to check that the function $\gamma^{\mathbb{T},\mathbb{T}_n}(t)$ is rd-continuous on $\mathbb{T}$. Also, there exists $t_n^* = t_n^*(t) \in \mathbb{T}_n$ satisfying

$$|t - t_n^*| = d(t, \mathbb{T}_n) = \inf\{|t - s| : s \in \mathbb{T}_n\}. \tag{3.4}$$

We choose $t_n^* = \gamma^{\mathbb{T},\mathbb{T}_n}(t)$ if $|t - \gamma^{\mathbb{T},\mathbb{T}_n}(t)| = d(t, \mathbb{T}_n)$, otherwise $t_n^* = \rho_n(\gamma^{\mathbb{T},\mathbb{T}_n}(t))$.

Fix $b \in \mathbb{T}$ and let $f : \widehat{\mathbb{T}} \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \widehat{\mathbb{T}} \times \mathbb{R}^d \to \mathbb{R}^d$ be two Borel functions satisfying the conditions

Assumption 3.1. *(Lipschitz condition) There exist a positive constant $K(b)$ such that for all $(t, x) \in [a, b]_{\widehat{\mathbb{T}}} \times \mathbb{R}^d$*

$$\|f(t, x) - f(t, y)\|^2 \vee \|g(t, x) - g(t, y)\|^2 \leqslant K\|x - y\|^2. \tag{3.5}$$

Assumption 3.2. *(Linear growth rate condition) There exist a positive constant $\overline{K} = \overline{K}(b)$ such that for all $(t, x) \in [a, b]_{\widehat{\mathbb{T}}} \times \mathbb{R}^d$*

$$\|f(t, x)\|^2 \vee \|g(t, x)\|^2 \leqslant \overline{K}(1 + \|x\|^2). \tag{3.6}$$

Assumption 3.3. *(Uniform continuty condition) The function $f(\cdot, x)$ and $g(\cdot, x)$ are continuous on $[a, b]_{\widehat{\mathbb{T}}}$, uniformly in $x \in \mathbb{R}^d$.*

On time scale $\mathbb{T}_n$, we consider the stochastic dynamic equation

$$\begin{cases} d^{\nabla_{\mathbb{T}_n}} x_n(t) = f(t, x_n(t_-))d^{\nabla_{\mathbb{T}_n}} t + g(t, x_n(t_-))d^{\nabla_{\mathbb{T}_n}} M(t), \quad t \in \mathbb{T}_n(a), \\ x(a) = \xi, \end{cases} \tag{3.7}$$

respectively on the time scale $\mathbb{T}$,

$$\begin{cases} d^{\nabla} x(t) = f(t, x(t_-))d^{\nabla_{\mathbb{T}}} t + g(t, x(t_-))d^{\nabla_{\mathbb{T}}} M(t), \quad t \in \mathbb{T}(a), \\ x(a) = \xi, \end{cases} \tag{3.8}$$

where ξ is an $\mathcal{F}_a$-random variable and $\mathbb{E}\,\|\xi\|^2 < \infty$. Under the assumptions 3.1 and 3.2, the equation (3.7)), has a a unique solution $x_n(\cdot)$ on $[a, b_n^*]_{\mathbb{T}_n}$. That is, $f(\cdot, x_n(\cdot_-))$ is ∇-integrable on $[a, b]_{\mathbb{T}_n}$; $\mathbb{E}\int_a^b \|g(s, x_n(s_-)\|^2\, \nabla t < \infty$ and

$$x_n(t) = \xi + \int_a^t f(s, x_n(s_-))\nabla_{\mathbb{T}_n} s + \int_a^t g(s, x_n(s_-))\nabla_{\mathbb{T}_n} M(s), t \in \mathbb{T}_n(a) \tag{3.9}$$

Respectively, the equation (3.8) has a unique solution $x(\cdot)$

$$x(t) = \xi + \int_a^t f(s, x(s_-))\nabla_{\mathbb{T}} s + \int_a^t g(s, x(s_-))\nabla_{\mathbb{T}} M(s), t \in \mathbb{T}(a). \tag{3.10}$$

Further, for any $a < b \in \widehat{\mathbb{T}}$ by (2.3) we have

$$\mathbb{E}\Big(\sup_{t\in[a,b]_{\mathbb{T}_n}} \|x_n(t)\|^2\Big) \leqslant (1 + 3\mathbb{E}\xi^2)e_\alpha(\widehat{\mathbb{T}}; a, T) \tag{3.11}$$

$$\mathbb{E}\Big(\sup_{t\in[a,b]_{\mathbb{T}}} \|x(t)\|^2\Big) \leqslant (1 + 3\mathbb{E}\xi^2)e_\alpha(\widehat{\mathbb{T}}; a, T), \tag{3.12}$$

where $\alpha = 3K(b - a + 4\hbar)$ (see: [6] for details).

In order to transform a function defined on $\mathbb{T}_n$ to a function defined on every time scale $\mathbb{T}$ we

$$f_n(t, x) = f(\gamma^{\mathbb{T},\mathbb{T}_n}(t), x);\;\; g_n(t, x) = g(\gamma^{\mathbb{T},\mathbb{T}_n}(t), x) \quad t \in \mathbb{T}; x \in \mathbb{R}^d, \tag{3.13}$$

$$h_n(t, x) = \|f(t, x) - f_n(t, x)\|^2 + \|g(t, x) - g_n(t, x)\|^2 \tag{3.14}$$

$$\widetilde{x}_n(t) = x(\gamma^{\mathbb{T},\mathbb{T}_n}(t)), \quad t \in \mathbb{T}. \tag{3.15}$$

To study the convergence of the solution sequence $x_n(t)$ of the IVPs (3.7) when $\mathbb{T}_n$ tends to $\mathbb{T}$, first assume that $\mathbb{T}_n \subset \mathbb{T}$. By the definition of Lebesgue nabla integral and stochastic nabla integral on time scales, we have

$$\int_a^t f(s, x(s_-))\nabla_{\mathbb{T}_n} s = \int_a^t f_n(s, \widetilde{x}_n(s_-))\nabla_{\mathbb{T}} s,$$

and

$$\int_a^t g(s, x(s_-))\nabla_{\mathbb{T}_n} M(s) = \int_a^t g_n(s, \widetilde{x}_n(s_-))\nabla_{\mathbb{T}} M(s),$$

for any $t \in \mathbb{T}_n$ (see, e.g. [4, 5]).

Lemma 3.1. *Let $x_n(t), n = 1, 2, \dots$ be solutions of the* (3.7) *and $x(t)$ be the solution of the* (3.8)*. Assume that $\mathbb{T}_n \subset \mathbb{T}$. Then, we have*

$$\mathbb{E}\|x(t) - x_n(t)\|^2 \leqslant \delta_b^{(n)} e_{K\beta}(\mathbb{T}; t, a),\; t \in [a, b]_{\mathbb{T}_n}, \tag{3.16}$$

and for all $t \in [a, b]_{\mathbb{T}}$,

$$\mathbb{E}\|x(t) - x_n(t_n^*)\|^2 \leqslant 2\delta_b^{(n)} e_{K\beta}(\mathbb{T}; t_n^*, a) + 2\Gamma_b(d_H(\mathbb{T}, \mathbb{T}_n) + \hbar)d_H(\mathbb{T}, \mathbb{T}_n). \tag{3.17}$$

where K is defined by (3.5)*, t_n^* is given by* (3.4)*, $\beta = 4(b - a + \hbar)$ and*

$$\delta_b^{(n)} = \beta\mathbb{E}\int_a^b \Big(\|f(s, x(s_-)) - f_n(s, \widetilde{x}_n(s_-))\|^2 + \|g(s, x(s_-)) - g_n(s, \widetilde{x}_n(s_-))\|^2\Big)\nabla_{\mathbb{T}} s. \tag{3.18}$$

and

$$\Gamma_b = 2\overline{K}[1 + (1 + 3\mathbb{E}\xi^2)e_\alpha(\mathbb{T}; b, a)]. \tag{3.19}$$

Proof. For any $t \in [a,b]_{\mathbb{T}_n}$, we have

$$\begin{aligned}\mathbb{E}\|x(t)-x_n(t)\|^2 \leqslant 2\mathbb{E}\left\|\int_a^t f(s,x(s_-))\nabla_{\mathbb{T}} s-\int_a^t f(s,x_n(s_-))\nabla_{\mathbb{T}_n} s\right\|^2 \\ +2\mathbb{E}\left\|\int_a^t g(s,x(s_-))\nabla_{\mathbb{T}} M(s)-\int_a^t g(s,x_n(s_-))\nabla_{\mathbb{T}_n} M(s)\right\|^2.\end{aligned} \tag{3.20}$$

It is seen that

$$\begin{aligned}&\mathbb{E}\left\|\int_a^t f(s,x(s_-))\nabla_{\mathbb{T}} s-\int_a^t f(s,x_n(s_-))\nabla_{\mathbb{T}_n} s\right\|^2 \\ &\leqslant 2\mathbb{E}\left\|\int_a^t f(s,x(s_-))\nabla_{\mathbb{T}} s-\int_a^t f(s,x(s_-))\nabla_{\mathbb{T}_n} s\right\|^2+2\mathbb{E}\left\|\int_a^t \left(f(s,x(s_-))-f(s,x_n(s_-))\right)\nabla_{\mathbb{T}_n} s\right\|^2 \\ &\leqslant 2(b-a)\mathbb{E}\int_a^t \|f(s,x(s_-))-f_n(s,\widetilde{x}_n(s_-))\|^2\,\nabla_{\mathbb{T}} s+2(b-a)K\mathbb{E}\int_a^t \|x(s_-)-x_n(s_-)\|^2\,\nabla_{\mathbb{T}_n} s,\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}&\mathbb{E}\left\|\int_a^t g(s,x(s_-))\nabla_{\mathbb{T}} M(s)-\int_a^t g(s,x_n(s_-))\nabla_{\mathbb{T}_n} M(s)\right\|^2 \\ &\qquad\leqslant 2\mathbb{E}\left\|\int_a^t g(s,x(s_-))\nabla_{\mathbb{T}} M(s)-\int_a^t g(s,x_n(s_-))\nabla_{\mathbb{T}_n} M(s)\right\|^2 \\ &\qquad\quad+2\mathbb{E}\left\|\int_a^t \left(g(s,x(s_-))-g(s,x_n(s_-))\right)\nabla_{\mathbb{T}_n} M(s)\right\|^2 \\ &\leqslant 2\hbar\mathbb{E}\int_a^t \|g(s,x(s_-))-g_n(s,\widetilde{x}_n(s_-))\|^2\,\nabla_{\mathbb{T}}\langle M\rangle_s+K\mathbb{E}\int_a^t \|x(s_-)-x_n(s_-)\|^2\,\nabla_{\mathbb{T}_n}\langle M\rangle_s \\ &\overset{(3.2)\text{ and }(3.3)}{\leqslant} 2\hbar\mathbb{E}\int_a^t \|g(s,x(s_-))-g_n(s,\widetilde{x}_n(s_-))\|^2\,\nabla_{\mathbb{T}} s+2\hbar K\mathbb{E}\int_a^t \|x(s_-)-x_n(s_-)\|^2\,\nabla_{\mathbb{T}_n} s.\end{aligned} \tag{3.22}$$

By substituting (3.21) and (3.22) in to (3.20), we obtain

$$\mathbb{E}\|x(t)-x_n(t)\|^2 \leqslant \delta_b^{(n)}+\beta K\int_a^t \mathbb{E}\|x(s_-)-x_n(s_-)\|^2\nabla_{\mathbb{T}_n} s. \tag{3.23}$$

From (3.23), using Gronwall-Bellman inequality in Lemma 2.2 and (2.3) obtain (3.16).

If $t \in \mathbb{T}$ then

$$\mathbb{E}\|x(t)-x_n(t_n^*)\|^2 \leqslant 2\mathbb{E}\|x(t)-x(t_n^*)\|^2+2\mathbb{E}\|x(t_n^*)-x_n(t_n^*)\|^2. \tag{3.24}$$

We have

$$\begin{aligned}\mathbb{E}\|x(t)-x(t_n^*)\|^2 &\leqslant 2\mathbb{E}\left\|\int_{t_n^*}^t f(x(s_-))\nabla_{\mathbb{T}} s\right\|^2+2\mathbb{E}\left\|\int_{t_n^*}^t g(x(s_-))\nabla_{\mathbb{T}} M(s)\right\|^2 \\ &\leqslant \Gamma_b(|t-t_n^*|+\hbar)|t-t_n^*| \leqslant \Gamma_b(d_H(\mathbb{T},\mathbb{T}_n)+\hbar)d_H(\mathbb{T},\mathbb{T}_n).\end{aligned} \tag{3.25}$$

Combining (3.24), (3.25) and using (3.16) obtain (3.17). The proof is complete. □

Lemma 3.2. *Assume that* $\mathbb{T}_n \subset \mathbb{T}$. *For each* ϵ *and* $b \in \mathbb{T}(a)$, *there exists* $\theta = \theta(\epsilon, T)$ *such that if* $d_H(\mathbb{T}, \mathbb{T}_n) < \theta$ *then*

$$\delta_b^{(n)} \leqslant \epsilon(b-a) + 4\beta\Big(2\Gamma_b + \frac{K\Gamma_b}{\overline{K}} - 2K\Big)\frac{b-a}{\theta} d_H(\mathbb{T}, \mathbb{T}_n), \tag{3.26}$$

where $\delta_b^{(n)}$ *is defined by* (3.18).

Proof. By Assumption 3.3, for each ϵ, there exists $\delta = \delta(\epsilon)$ such that if $t_1, t_2 \in [a, b]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$ and $x \in \mathbb{R}^d$ then

$$\|f(t_1, x) - f(t_2, x)\|^2 + \|g(t_1, x) - g(t_2, x)\|^2 \leqslant \frac{\varepsilon}{2\beta}. \tag{3.27}$$

Let $\overline{\Gamma}_b = (b - a + \hbar)\Gamma_b$. Choose $\theta = \theta(\epsilon, b) = \min\big\{\frac{\epsilon}{8\overline{\Gamma}_b K\beta}, \delta\big\}$. If $\gamma^{\mathbb{T},\mathbb{T}_n}(t) - t < \theta$ then

$$\begin{aligned}
&\mathbb{E}\|x(t) - x(\gamma^{\mathbb{T},\mathbb{T}_n}(t))\|^2 \\
&\quad\leqslant 2\mathbb{E}\left\|\int_t^{\gamma^{\mathbb{T},\mathbb{T}_n}(t)} f(s, x(s_-))\nabla_{\mathbb{T}} s\right\|^2 + 2\mathbb{E}\left\|\int_t^{\gamma^{\mathbb{T},\mathbb{T}_n}(t)} g(s, x(s_-))\nabla_{\mathbb{T}} M(s)\right\|^2 \\
&\qquad\leqslant 2\overline{K}(\theta + \hbar)\left[1 + (1 + 3\mathbb{E}\xi^2)e_\alpha(\mathbb{T}; b, a)\right](\gamma^{\mathbb{T},\mathbb{T}_n}(t) - t) < \overline{\Gamma}_b\theta \leqslant \frac{\varepsilon}{8K\beta}.
\end{aligned} \tag{3.28}$$

We see that the number of values $s \in [a, b]_{\mathbb{T}_n}$ and

$$\{t \in \mathbb{T} : \rho_n(s) < t < s, s - t \geqslant \theta\} \neq \emptyset,$$

is less than or equal to $[\frac{b-a}{\theta}]$. Assume that these values are $s_1 < s_2 < \ldots < s_r$ with $r \leqslant [\frac{b-a}{\theta}]$. In case $d_H(\mathbb{T}, \mathbb{T}_n) < \theta$, we see that if $t \in \mathbb{T}$ such that $\rho_n(s_i) < t < s_i, s_i - t \geqslant \theta$ then

$$t - \rho_n(s_i) = d(t, \mathbb{T}_n) \leqslant d_H(\mathbb{T}, \mathbb{T}_n).$$

Let

$$\tau_i = \max\{t \in \mathbb{T} : \rho_n(s_i) < t < s_i, s_i - t \geqslant \theta\}.$$

It is clear $\tau_i - \rho_n(s_i) \leqslant d_H(\mathbb{T}, \mathbb{T}_n)$. It is implied from (3.5) and (3.18) that

$$\begin{aligned}
\frac{\delta_b^{(n)}}{\beta} &= \mathbb{E}\int_a^b \Big(\|f(s, x(s_-)) - f_n(s, \widetilde{x}_n(s_-))\|^2 + \|g(s, x(s_-)) - g_n(s, \widetilde{x}_n(s_-))\|^2\Big)\nabla_{\mathbb{T}} s \\
&\leqslant 2\mathbb{E}\int_a^b \Big(\|f(s, x(s_-)) - f(s, \widetilde{x}_n(s_-))\|^2 + \|f(s, \widetilde{x}_n(s_-)) - f(\gamma^{\mathbb{T},\mathbb{T}_n}(s), \widetilde{x}_n(s_-))\|^2\Big)\nabla_{\mathbb{T}} s \\
&\quad + 2\mathbb{E}\int_a^b \Big(\|g(s, x(s_-)) - g(s, \widetilde{x}_n(s_-))\|^2 + \|g(s, \widetilde{x}_n(s_-)) - g(\gamma^{\mathbb{T},\mathbb{T}_n}(s), \widetilde{x}_n(s_-))\|^2\Big)\nabla_{\mathbb{T}} s \\
&\qquad \leqslant 4K\mathbb{E}\int_a^b \|x(s_-) - \widetilde{x}_n(s_-)\|^2\nabla_{\mathbb{T}} s + 2\mathbb{E}\int_a^b h_n(s, \widetilde{x}_n(s_-))\nabla_{\mathbb{T}} s
\end{aligned} \tag{3.29}$$

where $h_n(t,x)$ is defined by (3.14). We have

$$\mathbb{E}\int_a^b h_n(s,\widetilde{x}_n(s_-))\nabla_{\mathbb{T}}s = \mathbb{E}\int_a^{\rho_n(s_1)} h_n(s,\widetilde{x}_n(s_-))\nabla_{\mathbb{T}}s \\ + \sum_{i=1}^{r-1}\mathbb{E}\int_{\tau_i\wedge b}^{\rho_n(s_{i+1})\wedge b} h_n(s,\widetilde{x}_n(s_-))\nabla_{\mathbb{T}}s + \sum_{i=1}^{r}\mathbb{E}\int_{\rho_n(s_i)\wedge b}^{\tau_i\wedge b} h_n(s,\widetilde{x}_n(s_-))\nabla_{\mathbb{T}}s \quad (3.30)$$

and

$$\mathbb{E}\int_a^b \|x(s_-)-x(\gamma^{\mathbb{T},\mathbb{T}_n}(s_-))\|^2\nabla_{\mathbb{T}}s = \mathbb{E}\int_a^{\rho_n(s_1)} \|x(s_-)-x(\gamma^{\mathbb{T},\mathbb{T}_n}(s_-))\|^2\nabla_{\mathbb{T}}s \\ + \sum_{i=1}^{r-1}\mathbb{E}\int_{\tau_i\wedge b}^{\rho_n(s_{i+1})\wedge b} \|x(s_-)-x(\gamma^{\mathbb{T},\mathbb{T}_n}(s_-))\|^2\nabla_{\mathbb{T}}s \\ + \sum_{i=1}^{r}\mathbb{E}\int_{\rho_n(s_i)\wedge b}^{\tau_i\wedge b} \|x(s_-)-x(\gamma^{\mathbb{T},\mathbb{T}_n}(s_-))\|^2\nabla_{\mathbb{T}}s. \quad (3.31)$$

For all $t\in(a,\rho_n(s_1)]\bigcup\limits_{i=1}^{r-1}(\tau_i,\rho_n(s_{i+1})]$ we have

$$\mathbb{E}\|x(t)-x(\gamma^{\mathbb{T},\mathbb{T}_n}(t))\|^2 \leqslant \frac{\varepsilon}{8K\beta};$$

and by (3.27),

$$\mathbb{E}h_n(t,\widetilde{x}(t_-)) \\ = \mathbb{E}\big(\|f(t,\widetilde{x}(t_-))-f(\gamma^{\mathbb{T},\mathbb{T}_n}(t),\widetilde{x}(t_-))\|^2 + \|g(t,\widetilde{x}(t_-))-g(\gamma^{\mathbb{T},\mathbb{T}_n}(t),\widetilde{x}(t_-))\|^2\big) \leqslant \frac{\varepsilon}{4\beta}.$$

It follows that

$$\mathbb{E}\int_a^{\rho_n(s_1)} \|x(s_-)-x(\gamma^{\mathbb{T},\mathbb{T}_n}(s_-))\|^2\nabla_{\mathbb{T}}s \leqslant \frac{\varepsilon}{8K\beta}(\rho_n(s_1)-a) \leqslant \frac{\varepsilon}{8K\beta}(\tau_1-a),$$

$$\mathbb{E}\int_{\tau_i\wedge b}^{\rho_n(s_{i+1})\wedge b} \|x(s_-)-x(\gamma^{\mathbb{T},\mathbb{T}_n}(s_-))\|^2\nabla_{\mathbb{T}}s \leqslant \frac{\varepsilon}{8K\beta}(\rho_n(s_{i+1})\wedge b-\tau_i\wedge b) \\ \leqslant \frac{\varepsilon}{8K\beta}(\tau_{i+1}\wedge b-\tau_i\wedge b),$$

$$\mathbb{E}\int_a^{\rho_n(s_1)} h_n(s,\widetilde{x}_n(s_-))\nabla_{\mathbb{T}}s \leqslant \frac{\varepsilon}{4\beta}(\rho_n(s_1)-a) \leqslant \frac{\varepsilon}{4\beta}(\tau_1-a),$$

$$\mathbb{E}\int_{\tau_i\wedge b}^{\rho_n(s_{i+1})\wedge b} h_n(s,\widetilde{x}_n(s_-))\nabla_{\mathbb{T}}s \leqslant \frac{\varepsilon}{4\beta}(\rho_n(s_{i+1})\wedge b-\tau_i\wedge b) \leqslant \frac{\varepsilon}{4\beta}(\tau_{i+1}\wedge b-\tau_i\wedge b),$$

On the other hand, for $i = 1, 2, ..., r$ we have

$$\mathbb{E}\int_{\rho_n(s_i)\wedge b}^{\tau_i\wedge b} \|x(s_-) - x(\gamma^{\mathbb{T},\mathbb{T}_n}(s_-))\|^2 \nabla_{\mathbb{T}} s \leqslant \Big(\frac{\Gamma_b}{\overline{K}} - 2\Big)(\tau_i \wedge b - \rho_n(s_i)\wedge b)$$
$$\leqslant \Big(\frac{\Gamma_b}{\overline{K}} - 2\Big) d_H(\mathbb{T}, \mathbb{T}_n).$$

and by

$$\int_{\rho_n(s_i)\wedge b}^{\tau_i\wedge b} \mathbb{E} h_n(s, \widetilde{x}_n(s_-))\nabla_{\mathbb{T}} s \leqslant 4\Gamma_b(\tau_i \wedge b - \rho_n(s_i)\wedge b) \leqslant 4\Gamma_b d_H(\mathbb{T}, \mathbb{T}_n).$$

Thus, we obtain

$$\delta_b^{(n)} \leqslant \epsilon(\tau_1 - a) + \sum_{i=1}^{r-1} \epsilon(\tau_{i+1}\wedge b - \tau_i \wedge b) + \sum_{i=1}^{r} 4\beta\Big(2\Gamma_b + \frac{K\Gamma_b}{\overline{K}} - 2K\Big) d_H(\mathbb{T}, \mathbb{T}_n)$$
$$= \epsilon(b-a) + 4\beta\Big(2\Gamma_b + \frac{K\Gamma_b}{\overline{K}} - 2K\Big) r d_H(\mathbb{T}, \mathbb{T}_n)$$
$$\leqslant \epsilon(b-a) + 4\beta\Big(2\Gamma_b + \frac{K\Gamma_b}{\overline{K}} - 2K\Big)\frac{b-a}{\theta} d_H(\mathbb{T}, \mathbb{T}_n).$$

The proof is complete. □

We are now derive the convergence theorem for the IVPs (3.8) and (3.7).

Theorem 3.3. *Suppose that the sequence of time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ satisfy $\lim_{n\to\infty} \mathbb{T}_n = \mathbb{T}$. Suppose that assumption 3.1, 3.2 are satisfied. Let $x_n(t)$, $n = 1, 2, \dots$ be solutions of the* (3.8) *and $x(t)$ be the solution of the* (3.7)*. Then for any $T > a$, there holds*

$$\lim_{n\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \mathbb{E}|x(t) - x_n(t_n^*)|^2 = 0. \tag{3.32}$$

where t_n^* is defined by (3.4)*.*

Proof. Firstly, we assume that $\mathbb{T}_n \subset \mathbb{T}$ for all $n \in \mathbb{N}$. From Lemma 3.1, it follows that

$$\mathbb{E}\|x(t) - x_n(t_n^*)\|^2 \leqslant 2\delta_b^{(n)} e_\beta(\mathbb{T}_n; t_n^*, a) + 2\Gamma_a(d_H(\mathbb{T}, \mathbb{T}_n) + \hbar) d_H(\mathbb{T}, \mathbb{T}_n),$$

for all $t \in [a, b]_{\mathbb{T}}$. By Lemma 3.2, we get $\lim_{n\to\infty} \delta_b^{(n)} = 0$. Therefore, (3.32) holds. In the general case, we put

$$\widehat{\mathbb{T}}_n = \mathbb{T}_n \cup \mathbb{T}.$$

Then, it is easy to see that

$$d_H(\mathbb{T}, \mathbb{T}_n) = \max\{d_H(\widehat{\mathbb{T}}_n, \mathbb{T}), d_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n)\}. \tag{3.33}$$

Let $\widehat{x}_n(t)$ is the solution of equation (3.10) on time scale $\widehat{\mathbb{T}}_n$. For $t \in \mathbb{T}$, we have

$$\mathbb{E}\|x(t) - x_n(t_n^*)\|^2 \leqslant 2\mathbb{E}\|\widehat{x}_n(t) - x(t)\|^2 + 2\mathbb{E}\|\widehat{x}_n(t) - x_n(t_n^*)\|^2.$$

By Lemma 3.1, for $\mathbb{T} \subset \widehat{\mathbb{T}}_n$ and $\mathbb{T}_n \subset \widehat{\mathbb{T}}_n$, we imply

$$\mathbb{E}\|\widehat{x}_n(t) - x(t)\|^2 \leqslant \widehat{\delta}_b^{(n1)} e_\beta(\mathbb{T}; t, a),$$

$$\mathbb{E}\|\widehat{x}_n(t) - x_n(t_n^*)\|^2 \leqslant \widehat{2}\delta_b^{(n2)} e_\beta(\mathbb{T}_n; t, a) + 2\Gamma_b(d_H(\mathbb{T}, \mathbb{T}_n) + \hbar) d_H(\mathbb{T}, \mathbb{T}_n),$$

where

$$\begin{aligned} \widehat{\delta}_b^{(n1)} = \beta\mathbb{E}\int_a^b \Big(&\|f(s, x(s_-)) - f(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}}(s), x(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}}(s_-)))\|^2 \\ &+ \|g(s, x(s_-)) - g(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}}(s), x(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}}(s_-)))\|^2\Big) \nabla_{\widehat{\mathbb{T}}_n} s \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \widehat{\delta}_b^{(n2)} = \beta\mathbb{E}\int_a^b \Big(&\|f(s, x(s_-)) - f(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}_n}(s), x(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}_n}(s_-)))\|^2 \\ &+ \|g(s, x(s_-)) - g(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}_n}(s), x(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}_n}(s_-)))\|^2\Big) \nabla_{\widehat{\mathbb{T}}_n} s \end{aligned} \tag{3.35}$$

By the similar argument of Lemma 3.2 we can see that $\widehat{\delta}_{n1} \to 0, \widehat{\delta}_{n2} \to 0$ as $n \to \infty$. Thus, (3.32) holds. The proof is complete. □

For estimating the convergent rate, we need the following lemma.

Lemma 3.4 ([10]). *Assume that $\mathbb{T}_n \subset \mathbb{T}$. Then, we have*

$$\int_a^b (s - \gamma^{\mathbb{T}, \mathbb{T}_n}(s)) \nabla_{\mathbb{T}} s \leqslant 2(b-a) d_H(\mathbb{T}, \mathbb{T}_n).$$

Assume further that $f(t, x), g(t, x)$ satisfy the Lipschitz condition in both variables t and x, that is

$$\|f(t, x) - f(s, y)\|^2 \vee \|g(t, x) - g(s, y)\|^2 \leqslant \overline{\overline{K}}(|t-s|^2 + \|x-y\|^2), \tag{3.36}$$

for all $s, t \in \mathbb{T}$ and $x, y \in \mathbb{R}^d$. We now estimate the convergent rate of approximation.

Theorem 3.5. *Assume that assumption (3.36) is satisfied. Let $x_n(t)$, $n = 1, 2, \ldots$ be solutions of the (3.7) and $x(t)$ be the solution of the (3.8). If $t \in [a, b]_{\mathbb{T}}$ then*

$$\mathbb{E}\|x(t) - x_n(t_n^*)\|^2 \leqslant C_1 d_H(\mathbb{T}, \mathbb{T}_n), \tag{3.37}$$

where $C_1 = 8\overline{\overline{K}}[(\Gamma_b(b-a+\hbar) + b - a)(b-a) + 2\Gamma_b(d_H(\mathbb{T}, \mathbb{T}_n) + \hbar)]$ and t_n^* is defined by (3.4). Moreover, if $t \in \mathbb{T} \cap \mathbb{T}_n : a \leqslant t < b$ then

$$\mathbb{E}\|x(t) - x_n(t)\|^2 \leqslant C_2 d_H(\mathbb{T}, \mathbb{T}_n),$$

where $C_2 = 4\overline{\overline{K}}\beta[\Gamma_b(b-a+\hbar) + b - a](b-a)]$.

Proof. Let

$$\widehat{\mathbb{T}}_n = \mathbb{T}_n \cup \mathbb{T},$$

and $\widehat{X}_n(t)$ be the solution of equation (3.7) on the time scale $\widehat{\mathbb{T}}_n$. It is implied from (2.3) and Theorem 3.3, for $t \in [a,b]_{\mathbb{T}}$, we have

$$\begin{aligned}\mathbb{E}\|x(t) - x_n(t_n^*)\|^2 &\leqslant 2\mathbb{E}\|\widehat{x}_n(t) - x(t)\|^2 + 2\mathbb{E}\|\widehat{x}_n(t) - x_n(t_n^*)\|^2,\\ &\leqslant (\widehat{\delta}_b^{(n1)} + \widehat{\delta}_b^{(n2)})e^{\beta(t-a)} + 2\Gamma_b(d_H(\mathbb{T},\mathbb{T}_n) + \hbar)d_H(\mathbb{T},\mathbb{T}_n),\end{aligned}$$

where $\widehat{\delta}_b^{(n1)}, \widehat{\delta}_b^{(n2)}$ are given by (3.34)(3.35). Note that if $t \in \mathbb{T} \cap \mathbb{T}_n : a \leqslant t \leqslant b$ then

$$\mathbb{E}\|x(t) - x_n(t)\|^2 \leqslant (\widehat{\delta}_b^{(n1)} + \widehat{\delta}_b^{(n2)})e^{\beta(t-a)}.$$

Since $f(t,x), g(t,x)$ satisfy the Lipschitz condition (3.36),

$$\begin{aligned}\widehat{\delta}_b^{(n1)} &= \beta\mathbb{E}\int_a^b \|f(s,\widehat{x}_n(s)) - f(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s), \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)))\|^2 \nabla_{\widehat{\mathbb{T}}_n} s\\ &\quad + \beta\mathbb{E}\int_a^b \|g(s,\widehat{x}_n(s)) - g(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s), \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)))\|^2 \nabla_{\widehat{\mathbb{T}}_n} s\\ &\leqslant 2\overline{\overline{K}}\beta\mathbb{E}\int_a^b (|s - \gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)|^2 + \|\widehat{x}_n(s) - \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s))\|^2)\nabla_{\widehat{\mathbb{T}}_n} s.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\|\widehat{x}_n(s) - \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s))\|^2 &\leqslant 2\mathbb{E}\Big\| \int_{\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)}^s f(u,\widehat{x}_n(u_-))\nabla_{\widehat{\mathbb{T}}_n} u\Big\|^2\\ &\quad + 2\mathbb{E}\Big\| \int_{\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)}^s g(u,\widehat{x}_n(u_-)\nabla_{\widehat{\mathbb{T}}_n} M(u)\Big\|^2\\ &\leqslant \Gamma_b((s - \gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)) + \hbar)(s - \gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)).\end{aligned}$$

Therefore, by Lemma 3.4, we get

$$\begin{aligned}\widehat{\delta}_b^{(n1)} &\leqslant 2\overline{\overline{K}}[\Gamma_b(b-a+\hbar) + b - a]\int_a^b |s - \gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)|\nabla_{\widehat{\mathbb{T}}_n} s\\ &\leqslant 4\overline{\overline{K}}\beta[\Gamma_b(b-a+\hbar) + b - a](b-a)d_H(\widehat{\mathbb{T}}_n,\mathbb{T}).\end{aligned}$$

Similarly, we imply that

$$\widehat{\delta}_b^{(n2)} \leqslant 4\overline{\overline{K}}\beta[\Gamma_b(b-a+\hbar) + b - a](b-a)d_H(\widehat{\mathbb{T}}_n,\mathbb{T}_n).$$

Thus, we obtain

$$\begin{aligned}\|x(t) - x_n(t_n^*)\| &\leqslant (\widehat{\delta}_b^{(n1)} + \widehat{\delta}_b^{(n2)})e_\beta(\widehat{\mathbb{T}}_n; t, a) + 2\Gamma_b(d_H(\mathbb{T},\mathbb{T}_n) + \hbar)d_H(\mathbb{T},\mathbb{T}_n)\\ &\leqslant 8\overline{\overline{K}}\beta[\Gamma_b(b-a+\hbar) + b - a](b-a)d_H(\mathbb{T},\mathbb{T}_n)\\ &\quad + 2\Gamma_a(d_H(\mathbb{T},\mathbb{T}_n) + \hbar)d_H(\mathbb{T},\mathbb{T}_n) = C_1 d_H(\mathbb{T},\mathbb{T}_n),\end{aligned}$$

where $C_1 = 8\overline{\overline{K}}[(\Gamma_b(b-a+\hbar) + b - a)(b-a) + 2\Gamma_b(d_H(\mathbb{T},\mathbb{T}_n) + \hbar)]$. Similarly, if $t \in \mathbb{T} \cap \mathbb{T}_n : a \leqslant t < b$ then

$$\|x(t) - x_n(t)\| \leqslant C_2 d_H(\mathbb{T},\mathbb{T}_n),$$

where $C_2 = 4\overline{\overline{K}}\beta[\Gamma_b(b-a+\hbar) + b - a](b-a)]$. The proof is complete. □

4 Applications to the Euler-Maruyama method

In this section, we will first show that the convergence of solution of the Euler-Maruyama method for SDEs is a simple corollary by using results of previous section. After that we give some examples to illustrate the power of obtained results.

Consider the IVP

$$\begin{cases} dx(t) = f(t, x(t))dt + g(t, x(t))dM(t) \quad t \in [a, b] \\ x(a) = x_0, \end{cases} \tag{4.1}$$

In numerical analysis, approximations to the solution $x(t)$ of (4.1) will be generated at various values, called mesh points, in the interval $[a, b]$. For a positive integer n, we select a subdivision of the interval $[a, b]$

$$a = a^{(n)} < t_1^{(n)} < \cdots < t_{k_n-1}^{(n)} < t_{k_n}^{(n)} := b, \ k_n \in \mathbb{N}. \tag{4.2}$$

Associating with (4.2), we study a stochastic difference equation, called the Euler-Maruyama method [8, 14, 15] as follows

$$\begin{cases} x_0^{(n)} &= x_0, \\ x_i^{(n)} &= x_{i-1}^{(n)} + f(t_{i-1}^{(n)}, x_{i-1}^{(n)})(t_i^{(n)} - t_{i-1}^{(n)}) + g(t_{i-1}^{(n)}, x_{i-1}^{(n)})(M(t_i^{(n)}) - M(t_{i-1}^{(n)})), \end{cases} \tag{4.3}$$

for $i = 1, 2, \dots, k_n$. Let $\mathbb{T} := [a, b]$ and $\mathbb{T}_n := \{a^{(n)}, t_1^{(n)}, \dots, t_{k_n-1}^{(n)}, t_{k_n}^{(n)}\}$. Then $\mathbb{T}$ and $\mathbb{T}_n$ are time scales. Let $\hat{f}(t, \cdot) = f(t_-, \cdot)$ and $\hat{g}(t, \cdot) = g(t_-, \cdot)$. This leads to that we can rewrite (4.1) and (4.3) as follows

$$d^\nabla x(t) = \hat{f}(t, x(t_-))d^\nabla t + \hat{g}(t, x(t_-))d^\nabla M(t), \ t \in \mathbb{T}, x(a) = x_0,$$

and

$$d^{\nabla_n} x_n(t) = \hat{f}(t, x_n(t_-))d^{\nabla_n} t + \hat{g}(t, x_n(t_-))d^{\nabla_n} M(t), \ \tau \in \mathbb{T}_n, x_n(a) = x_0,$$

respectively. In this case, it is easy to see that

$$2d_H(\mathbb{T}, \mathbb{T}_n) = h_n := \sup_{1 \leqslant i \leqslant k_n} \{t_i^{(n)} - t_{i-1}^{(n)}\} \ \text{ for all } n \in \mathbb{N}. \tag{4.4}$$

Suppose that $f(t, x), g(t, x)$ are continuous and satisfy the Lipschitz conditions

$$\|f(t, x_1) - f(t, x_2)\| \leqslant k_1 \|x_1 - x_2\|, \ \text{for all } t \in [a, b],$$

$$\|g(t, x_1) - g(t, x_2)\| \leqslant k_2 \|x_1 - x_2\|, \ \text{for all } t \in [a, b].$$

Then $\hat{f}(t, x), \hat{g}(t, x)$ also satisfy the Lipschitz conditions

$$\|\hat{f}(t, x_1) - \hat{f}(t, x_2)\| \leqslant k_1 \|x_1 - x_2\|, \ \text{for all } t \in [a, b],$$

$$\|\hat{g}(t, x_1) - \hat{g}(t, x_2)\| \leqslant k_2 \|x_1 - x_2\|, \ \text{for all } t \in [a, b].$$

By Theorem 3.3 and (4.4), we see that $x_n(t)$ will mean-square converge to $x(t)$ on $[a,b]$, that is

$$\lim_{n\to\infty}\sup_{t\in[a,b]}\mathbb{E}|x(t)-x_n(t_n^*)|^2=0.$$

Hence, we obtain the well-known result for the convergence of the Euler-Maruyama method in numerical analysis [8, 14, 15].

Assume further that f,g satisfy the Lipschitz conditions in both variables with constant k_1,k_2. That is,

$$\|f(t,x_1)-f(s,x_2)\|\leqslant k_1(|t-s|+\|x_1-x_2\|),\ \text{for all}\ t,s\in[a,b],\ x_1,x_2\in\mathbb{R}^d,$$

$$\|g(t,x_1)-g(s,x_2)\|\leqslant k_2(|t-s|+\|x_1-x_2\|),\ \text{for all}\ t,s\in[a,b],\ x_1,x_2\in\mathbb{R}^d.$$

Then $\hat{f}(t,x),\hat{g}(t,x)$ also satisfy the Lipschitz conditions in both variables with constant k_1,k_2,

$$\|\hat{f}(t,x_1)-\hat{f}(s,x_2)\|\leqslant k_1(|t-s|+\|x_1-x_2\|),\ \text{for all}\ t,s\in[a,b],\ x_1,x_2\in\mathbb{R}^d,$$

$$\|\hat{g}(t,x_1)-\hat{g}(s,x_2)\|\leqslant k_2(|t-s|+\|x_1-x_2\|),\ \text{for all}\ t,s\in[a,b],\ x_1,x_2\in\mathbb{R}^d.$$

By Theorem 3.5, we get an estimation of the convergent rate as well as an error bound for the Euler-Maruyama method as follows

$$\sup_{0\leqslant i\leqslant k_n}\mathbb{E}\|x(t_i^{(n)})-x_i^{(n)}\|\leqslant Ch_n,\quad\text{for all}\ n\in\mathbb{N}.$$

We now consider some examples in particular case.

Example 4.1. We now, for all $n\in\mathbb{N}$, let

$$\mathbb{T}_n=\mathbb{P}_{n,\frac{1}{n}}=\bigcup_{k=0}^{\infty}\Big[k(n+\frac{1}{n}),k(n+\frac{1}{n})+\hbar\Big].$$

On $\mathbb{T}_n$, we consider ∇-equation

$$\begin{cases}d^{\nabla_n}x_n(t)=rx_n(t_-)d^{\nabla_n}t+px(t_-)d^{\nabla_n}W(t)\quad\forall\ t\in[0,T]\\ x(0)=1,\end{cases}\tag{4.5}$$

where r,p are two constants and $W=(W(t))$ is an one-dimension Brownian motion. From [6, Example 4.6] implies equation (4.5) has follow solution

$$X_t^{(n)}\ =\ e^{rt+pW_t-\frac{1}{2}p^2[W]_t+\sum\limits_{0<s\leqslant t}\frac{1}{2}p^2(\nabla^*W_s)^2}\times\prod_{s\leqslant t}(1\ +\ r\nu_n(s)\ +\ p\nabla^*W_s)e^{-r\nu_n(s)-p\nabla^*W_s}.\tag{4.6}$$

By using [20, Theorem 2.1] we have

$$X_t^{(n)}=e^{\frac{2r-p^2}{2}t+pW_t}\times\prod_{s\leqslant t}(1+\frac{r}{n}+p\nabla^*W_s)e^{\frac{p^2-2r}{2n}-p\nabla^*W_s}.\tag{4.7}$$

On $\mathbb{T} = \mathbb{R}_+$, we consider differential equation

$$\begin{cases} dx(t) = rx(t)dt + px(t)dW_t \quad \forall\, t \in [0, T] \\ x(0) = 1. \end{cases} \tag{4.8}$$

This equation has solution as following

$$x(t) = e^{\frac{2r-p^2}{2}t + pW_t}.$$

By calculating, we see that

$$\begin{aligned} &\lim \mathbb{E}\left(\prod_{s\leqslant t}(1 + r\nu_n(s) + p\nabla^* W_s) - \prod_{s\leqslant t}\left(e^{\frac{2r-p^2}{2}\nu_n(s) + p\nabla^* W_s}\right)\right)^2 \\ &= \lim_{n\to\infty}\left\{\prod_{s\leqslant t}\left((1 + r\nu_n(s))^2 + p^2\nu_n(s)\right) - 2\prod_{s\leqslant t}\left(e^{r\nu(s)}\left[1 + r\nu_n(s) + p^2\nu_n(t)\right]\right)\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\left. + \prod_{s\leqslant t} e^{(2r+p^2)\nu_n(s)}\right\} = 0. \end{aligned}$$

This implies that

$$\lim_{n\to\infty}\mathbb{E}|X^{(n)}(t) - x(t)|^2 = 0.$$

5 Conclusion

In this paper, we have proved the convergence of solutions of stochastic dynamic equations on time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ when this sequence converges to a time scale $\mathbb{T}$. The convergent rate of solutions is estimated when the drift and diffussion coefficients satisfies the Lipschitz condition in both variables. By using these results, the convergence of the Euler-Maruyama method for SDEs is obtained as a corollary.

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